Gordian complexes of Knots given by region crossing change and arc shift moves

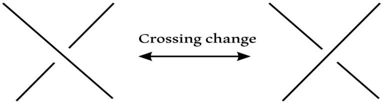
Amrendra Gill

(Joint work with Dr. M. Prabhakar and Prof. Andrei Vesnin)

Department of Mathematics

Indian Institute of Technology Ropar, India

Knots and Gordian Distance



Remark: Crossing change is an unknotting operation for classical knots.

For a pair of knots K and K', define

Definition[1]

Gordian distance $d_G(K, K')$ = minimal number of crossing changes required to transform a diagram of K into a diagram of K', where minimal is taken over all diagrams of K and K'.

(X, d_G) is a metric space, where X denotes the set of all isotopy classes of oriented knots in $\mathbb{S}^3$.

[1] H. Murakami, *Some metrics on classical knots*, Math. Ann. 270 (1985) 35–45.

Gordian complex $\mathcal{G}$ of knots (Hirasawa, Uchida 2002)

Definition[2]

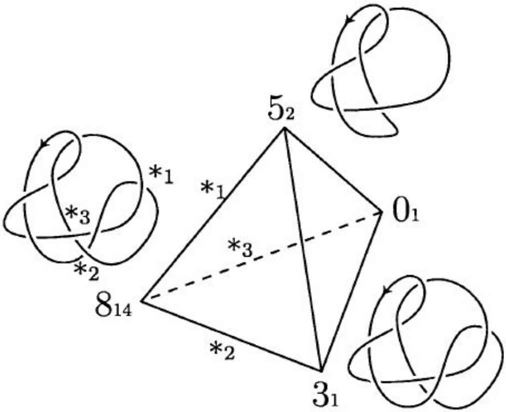
The Gordian complex $\mathcal{G}$ of knots is a simplicial complex defined by the following;

1. The vertex set of $\mathcal{G}$ consists of all the isotopy classes of oriented knots in $\mathbb{S}^3$, and
2. A family of n+1 vertices $\{K_0, K_1, ..., K_n\}$ spans an n-simplex if and only if the Gordian distance $d_G(K_i, K_j) = 1$ for any distinct members of the family.

[2] M. Hirasawa, Y. Uchida, *The Gordian complex of knots*, J. Knot Theory Ramifications **11** (2002), no. 3, 363–368.

An example of 3-simplex

Knots $\{0_1, 3_1, 5_2, 8_{14}\}$ spans an 3-simplex in $\mathcal{G}$.



Results for $\mathcal{G}$

Theorem[2]

For any 1-simplex e of the Gordian complex, there exists an infinitely high dimensional simplex σ such that e is contained in σ.

Corollary[2]

For any knot K_0, there exists an infinite family of knots K_0,K_1,K_2,...such that the Gordian distance $d_G(K_i, K_j) = 1$, for any $i \neq j$.

[2] M. Hirasawa, Y. Uchida, *The Gordian complex of knots*, J. Knot Theory Ramifications **11** (2002), no. 3, 363–368.

More study on Gordian complexes of knots

- Y. Ohyama(2006) defined and studied the C_k-Gordian complex of knots using a local move called C_k-move.[3]

- H(n)-Gordian complex of knots was explored by K. Zhang et al.(2017).[4]

- The Gordian complexes by pass-move and $\bar{\sharp}$-move were explored by K. Zhang and Z. Wang(2018).[5]

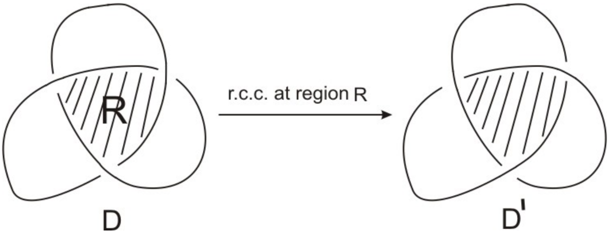
[3] Y. Ohyama, *The C_k Gordian complex of knots*, J. Knot Theory Ramifications 15 (2006) 73–80.

[4] K. Zhang, Z. Yang and F. Lei, *The H(n)-Gordian complex of knots*, J. Knot Theory Ramifications 26 (2017) 1750088.

[5] K. Zhang, Z. Wang, *A note on the Gordian complexes of some local moves on knots*, J. Knot Theory Ramifications **27** (2018), no. 9, 1842002.

Region crossing change(R.C.C.)

R.C.C. at region R = Switching all crossings lying at the boundary of R.
Example:



Theorem[6]

Every knot diagram can be transformed into a diagram of trivial knot by region crossing changes.

[6] A. Shimizu, *Region crossing change is an unknotting operation*, J. Math. Soc. Japan 66 (2014), no. 3, 693–708.

Gordian complex by r.c.c., $\mathcal{G}_R$

$d_R(K, K')$ = minimum no. of r.c.c. required to convert K into K', min. being taken over all diagrams (both minimal and non minimal) of K and K'.

Definition

A set of $n+1$ knots $\{K_0, K_1, ..., K_n\}$ spans an n-simplex in $\mathcal{G}_R$ iff $d_R(K_i, K_j) = 1$ for all $i \neq j$.

A. Gill, M. Prabhakar, A. Vesnin, *Gordian complexes of knots and virtual knots given by region crossing changes and arc shift moves*, J. Knot Theory Ramifications **29** (2020), 2042008(online available).

Gordian complex by r.c.c., $\mathcal{G}_R$

Question: Does there exists an m-simplex for all $m = 1, 2, 3, ...$ in $\mathcal{G}_R$?

Theorem

Given any knot K_0 and any $m \in \mathbb{N}$, there exists an m-simplex $\{K_0, K_1, ..., K_m\}$ containing K_0 in $\mathcal{G}_R$.

A. Gill, M. Prabhakar, A. Vesnin, *Gordian complexes of knots and virtual knots given by region crossing changes and arc shift moves*, J. Knot Theory Ramifications **29** (2020), 2042008(online available).

Proof of Theorem

Brief sketch:

- Considering K_0 as trivial knot, we construct an infinite family of knots $\{K_0, K_1, \ldots, \}$ such that $d_R(K_i, K_j) \leq 1$ for $i \neq j \geq 0$.
- $\{K_0, K_1, \ldots, \}$ need to be shown distinct.
- We use the polynomial invariant $c_0(L; x)$ of oriented links to distinguish K_i's.
- Polynomial $c_0(L; x)$ comes as one of the coefficient polynomial of HOMFLY polynomial.(A. Kawauchi, [7])
- Computations show that for distinct m, $c_0(K_m; x)$ have different degrees.
- Hence all the knots $\{K_0, K_1, \ldots, \}$ are distinct and Theorem follows.

∎

[7] A. Kawauchi, *On coefficient polynomials of the skein polynomial of an oriented link*, Kobe J. Math. 11(1) (1994) 49–68.

Gordian complex of virtual knots(S. Horiuchi et al., 2012)

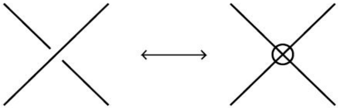


Figure: *v*-move

Note: v-move is an unknotting operation for virtual knots. So define,

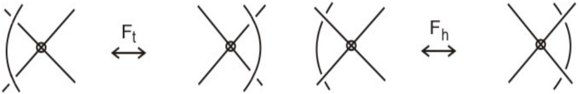
$d_v(K, K')$ = minimum no. of v-moves required to convert K into K', min. being taken over all diagrams of K and K'.

S. Horiuchi *et al.*[8] extended the notion of Gordian complex to virtual knots using distance $d_v(K, K')$ by *v*-move.

[8] S. Horiuchi, K. Komura, Y. Ohyama and M. Shimozawa, *The Gordian complex of virtual knots*, J. Knot Theory Ramifications 21(13) (2012) 1250122

Gordian complex by forbidden moves

Forbidden-moves are local moves in virtual knot diagrams defined as,



Note: forbidden moves are unknotting operation for virtual knots.[9]
Define,

$d_F(K, K')$ = minimum no. of forbidden moves required to convert a diagram of K into K'.

S. Horiuchi, Y. Ohyama(2013)[10] defined Gordian complex of virtual knots by forbidden moves using $d_F(K, K')$.

[9] S. Nelson, *Unknotting virtual knots with Gauss diagram forbidden moves*, J. Knot Theory Ramifications 10(6) (2001) 931–935

[10] S. Horiuchi, Y. Ohyama, *The Gordian complex of virtual knots by forbidden moves*, J. Knot Theory Ramifications **22** (2013), no. 9, 135051

Arc shift move

An arc in a virtual knot diagram D is defined as the segment passing through exactly one pair of crossings.

Arc shift move on an arc (a, b) is defined as follows,[11]

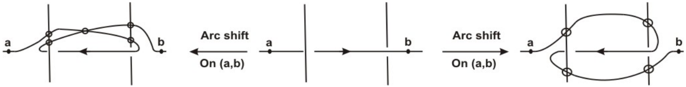


Figure: Arc shift on arc (a, b)

- Arc shift move with GR moves is an unknotting operation for virtual knots.[11]

[11] A. Gill, K. Kaur and M. Prabhakar *Arc shift number and region arc shift number for virtual knots*, J. Korean Math. Soc. **56**(4) (2019), 1063–1081..

Gordian complex by arc shift move

For any two virtual knots K_1, K_2 we define,

$$d_A(K_1, K_2) = \text{minimum no. of arc shift moves needed to convert a diagram of } K_1 \text{ into a diagram of } K_2.$$

Definition

The Gordian complex $\mathcal{G}_{\mathcal{A}}$ of virtual knots by arc shift move is a simplicial complex defined by the following;

1. Each virtual knot K is a vertex of $\mathcal{G}_{\mathcal{A}}$.
2. Any set $\{K_i\}_{i=0}^{n}$ of n+1 vertices spans an n-simplex if and only if $d_A(K_t, K_s) = 1$ for $t, s = 0, 1, ..., n$ such that $t \neq s$.

A. Gill, M. Prabhakar, A. Vesnin, *Gordian complexes of knots and virtual knots given by region crossing changes and arc shift moves*, J. Knot Theory Ramifications **29** (2020), 2042008(online available).

Example of a 2-simplex in $\mathcal{G}_{\mathcal{A}}$

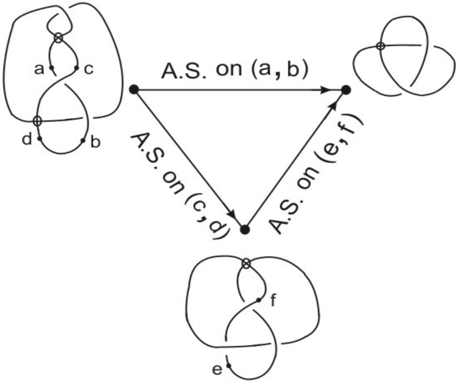


Figure: 2-simplex in $\mathcal{G}_{\mathcal{A}}$

Results for more simplexes in $\mathcal{G}_{\mathcal{A}}$

Theorem

For any $n \in \mathbb{N}$, there exists an n-simplex $\{VK_0, VK_1, \ldots, VK_n\}$ in $\mathcal{G}_{\mathcal{A}}$.

Corollary

Given any 0-simplex $\{K\}$ in $\mathcal{G}_{\mathcal{A}}$, there exists an n-simplex containing K for each positive integer n.

Proof of Theorem

Brief idea:

- Starting with trivial knot VK_0, we construct an infinite family of virtual knots $\{VK_n\}_{n\geq 0}$ s.t. $d_A(VK_t, VK_s) \leq 1$ for distinct $t, s \geq 0$.
- Establish $\{VK_n\}_{n\geq 0}$ as distinct virtual knots.
- We compute degree of Jones-Kauffman f-polynomial for VK_n and use it to distinguish virtual knots $VK_n's$.
- From computations it turns out that max. degree $f_{VK_n}(A) = 4n$.
- Since $f_{VK_n}(A)$ have different degree so $\{VK_n\}_{n\geq 0}$ are distinct virtual knots and Theorem follows.

Proof of the corollary

For any given virtual knot K consider the family,

$$\sigma_n = \{K\sharp VK_0, K\sharp VK_1, ..., K\sharp VK_n\}$$

where each $K\sharp VK_i$ is the connected sum of knots K and VK_i.

Since VK_0 is trivial, so $K\sharp VK_0 \backsim K$ and hence σ_n is an n-simplex containing K.

Problems Further

- For a given n-simplex τ_n for $n \geq 1$, is it possible to find simplexes τ_m of all dimensions $m > n$ such that $\tau_n \subseteq \tau_m$?(for $n = 1$ existence shown by Hirasawa and Uchida in the case of Gordian complex by crossing change)

Thank You