Around Birkhoff conjecture for Magnetic billiards

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Michael Bialy, Tel Aviv University

based on joint works with A.E. Mironov

Introduction

1. We confirm Anatole Katok's expression: Billiard table is a playground for mathematicians.

Attractions available on the playground:

Birkhoff billiards $\mathcal{B}$; Angular billiards $\mathcal{A}$; Magnetic billiards $\mathcal{M}$.

Other toys:

Outer billiards, Outer magnetic billiards, Andreev billiards...

2. Integrable billiards. Amazingly in all integrable examples, integrals are polynomial in momenta.

Integrals for the (A) Circle and (B) Ellipse:

In case A :

$$F(x, v) = xv_y - yv_x,$$

is the momentum of (x, v) with respect to 0.

In case B:

$$F(x, v) = ((x - c)v_y - yv_x)((x + c)v_y - yv_x)$$

is the product of two momenta with respect to the foci $(\pm c, 0))$ of the ellipse.

3. Algebraic version of Birkhoff-Poritsky conjecture: *The only billiards admitting polynomial in v integrals are Circles and Ellipses*

Papers

- Bialy, M., Mironov, A.E. Algebraic non-integrability of magnetic billiards. J. Phys. A 49 (2016), no. 45.
- Bialy, M., Mironov, A.E. Angular Billiard and Algebraic Birkhoff conjecture. Adv. Math. 313 (2017), 102–126.
- Bialy, M., Mironov, A.E. Algebraic Birkhoff conjecture for billiards on Sphere and Hyperbolic plane. J. Geom. Phys. 115 (2017), 150–156.
- Glutsyuk, A., Shustin, E. On polynomially integrable planar outer billiards and curves with symmetry property. arXiv:1607.07593
- Glutsyuk, A. On algebraically integrable Birkhoff and angular billiards. arXiv:1706.04030

Important motivational papers:

1. Bolotin, S. V. *Integrable Birkhoff billiards.* (Russian) Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1990, no. 2, 33-36

2. Tabachnikov, S. *On algebraically integrable outer billiards.* Pacific J. Math. 235 (2008), no. 1, 89-92.

3. Bialy, M. *On totally integrable magnetic billiards on constant curvature surface.* Electron. Res. Announc. Math. Sci. 19 (2012), 112-119.

4. Robnik, M., Berry, M.V. *Classical billiards in magnetic fields.* J. Phys. A 18 (1985), no. 9, 1361-1378.

5. Gutkin, E., Tabachnikov, S. *Billiards in Finsler and Minkowski geometries.* J. Geom. Phys. 40 (2002), no. 3–4, 277-301.

6. Tabachnikov, S. *Remarks on magnetic flows and magnetic billiards, Finsler metrics and a magnetic analog of Hilbert's fourth problem.* In Modern dynamical systems and applications, 233-250, Cambridge Univ. Press, Cambridge, 2004.

Similarity with Outer billiards

Outer billiard (Neumann, Moser, Tabachnikov)

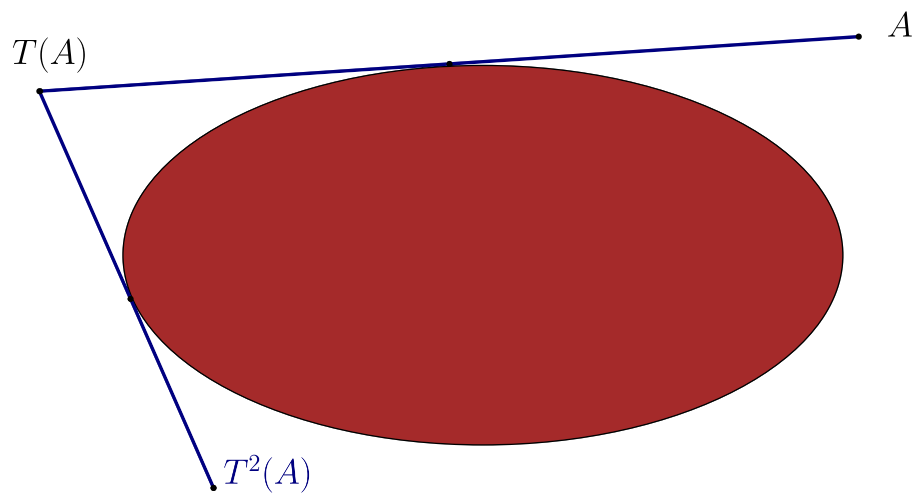

Figure 1: Outer billiard map.

Let γ is given by $\{F = 0\}$ then

$$F(x - \varepsilon F_y, y + \varepsilon F_x) = F(x + \varepsilon F_y, y - \varepsilon F_x).$$

Collect ε^3 terms:

$$F_{xxx}F_y^3 - 3F_{xxy}F_y^2F_x + 3F_{xyy}F_yF_x^2 - F_{yyy}F_x^3) = 0.$$

This is a complete derivative along the field $\left(F_y \frac{\partial}{\partial x} - F_x \frac{\partial}{\partial y}\right)$ of the affine Hessian

$$H(F) := F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2.$$

So

$$H(F) = const$$

Can be further studied by algebraic tools...(Tabachnikov, Glutsyuk-Shustin):

Theorem.

The only algebraically integrable outer billiards are ellipses.

Angular billiard

Birkhoff billiard $\mathcal{B}$ turns out to be Dual to **Angular billiard** $\mathcal{A}$.

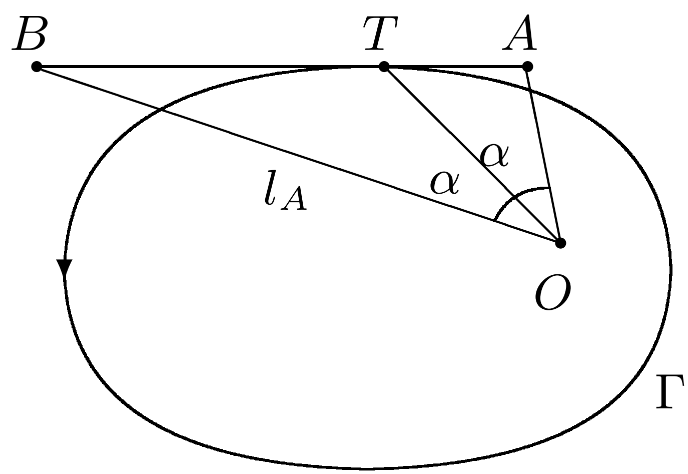

Figure.2 $\angle AOT = \angle TOB, \quad \mathcal{A}(A) = B.$

Let Γ the dual curve to γ consisting of points which are dual to the tangent lines of γ.

Theorem. *In a neidhborhood of the boundary curve Γ the Angular billiard is Dual to Birkhoff billiard inside γ. More precisely, if a line a is transformed to b by $\mathcal{B}$, then the for the dual points one has:* $\mathcal{A}(A) = B$.

Duality

Correspondence of polar duality with respect to the unit circle: It acts on *non-oriented* lines. Given a line l not passing through O we write l in the form $< n, x >= p,\ p > 0$, where n is an outward unit normal to l then the dual point corresponding to l is by definition $L = n/p$. In other words, the corresponding point L lies on the normal radius to the line at the distance equal $1/p$:

$$l = \{< n, x >= p,\ p > 0\} \leftrightarrow L = n/p.$$

Polar duality extends to the usual projective duality:

$$l = \{kx + ly = mz\} \ \leftrightarrow \ (k : l : m).$$

By small letters we denote the lines and by capital letters the corresponding dual points.

Duality preserves the incidence relation and dual to Γ is γ again. More precisely, if l is tangent to γ at Q then the dual line q is tangent to Γ at L (see Fig. 3).

Furthermore, if a, b are two oriented positive lines in the plane so that the Birkhoff billiard map $\mathcal{B}$ transforms a to b. Let $Q \in \gamma$ be the point of reflection and let l be the tangent line to γ at the reflection point Q. Then the dual points A, B lie on the line q which is tangent to Γ at L. Moreover, AOL and BOL are equal, so Angular billiard rule holds:

$$\mathcal{A}(A) = B.$$

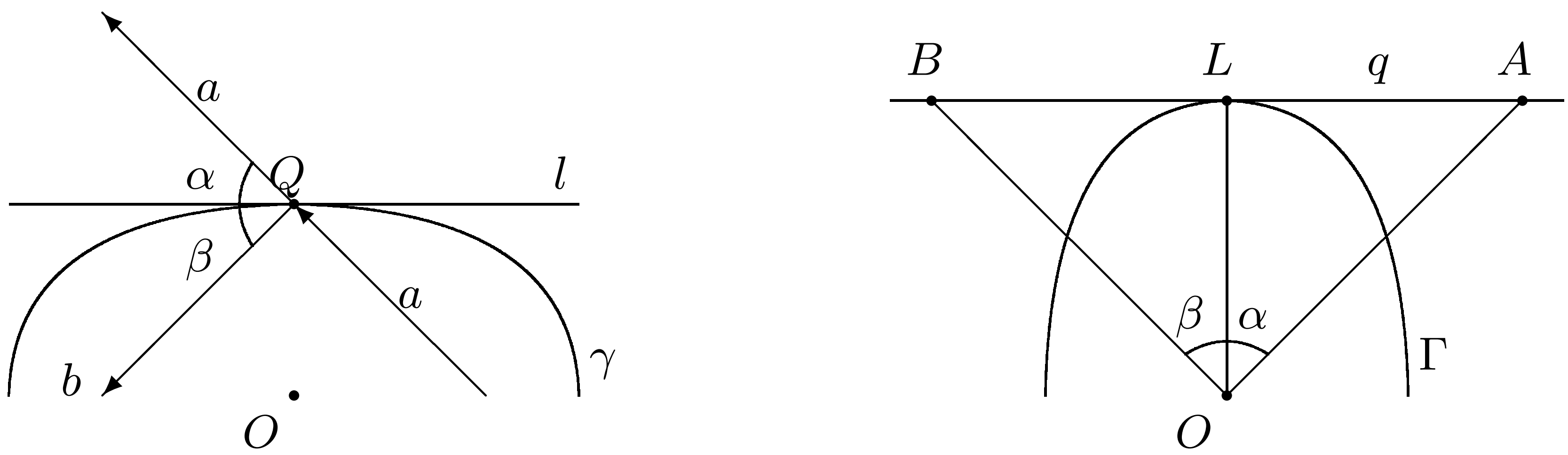

Figure 3 Polar duality; $\beta = \alpha$.

Remarkable equation-Angular billiard

Lemma. For the integral F of Angular billiard for Γ, for all small ε, and $(x, y) \in \Gamma$ we have:

(1)
$$F(x + \varepsilon F_y, y - \varepsilon F_x)\left(-\frac{\mu}{\varepsilon}\right)^{2m} = F(x + \mu F_y, y - \mu F_x),$$

$$\mu = -\frac{(x^2 + y^2)\varepsilon}{x^2 + y^2 + 2\varepsilon(xF_y - yF_x)}.$$

More complicated but somehow similar to the equation for Outer billiards. Can be treated by algebraic tools... Bialy-Mironov, Glutsyuk...Altogether one has the following

Theorem. (Algebraic Birkhoff conjecture)

Algebraic Birkhoff conjecture holds true: the existence of Polynomial in momenta integral for Birkhoff billiard implies that the boundary curve is an ellipse.

Magnetic billiards

Magnetic field of constant magnitude $\beta > 0$, the particle moves inside convex domain Ω with unit speed along Larmor circle of constant radius $r = \frac{1}{\beta}$ in a **counterclockwise** direction, hitting the boundary the particle is reflected according to the law of geometric optics. We shall assume that every smooth piece γ of the boundary of Ω satisfies

$$\beta < k_{\min} \qquad \Leftrightarrow \qquad r > \rho_{\max}$$

where k is the curvature (weak Magnetic field). Motivation: Berry-Robnik experiments: Magnetic billiard in ellipse is not integrable.

Beautiful picture by Peter Albers, Michael Hermann, Gautam Dilip Banhatti (with permission) of Magnetic billiards.

Example and Conjecture

Example. Let γ be a circle with the center at the origin. Then the function which measures the distance to the origin of the center of Larmor circle remains unchanged under the reflections and hence is the integral of the billiard flow. So the integral h has the form:

$$h(x,v) = x_1^2 + x_2^2 + \frac{2}{\beta}(v_1 x_2 - v_2 x_1).$$

Conjecture: The only integrable magnetic billiard is circular.

Here, as usual the integrability can be understood in various ways.
An approach of Hopf rigidity and *Total* integrability for magnetic billiards I reported here on 2013.

(2) $$\mathcal{L} : T_1\Omega \to \mathbb{R}^2,\ \mathcal{L}(x,v) = x + rJv,$$

which assigns to every unit tangent vector $v \in T_x\Omega$ the center of the corresponding Larmor circle. Varying unit vector v in $T_x\Omega$, for a fixed point $x \in \Omega$, the corresponding Larmor centers form a circle of radius r centered at x, and the domain swept by all these circles, when x runs over Ω, we shall denote by Ω_r. Vice versa, one can prove that for any circle of radius r lying in Ω_r its center necessarily belongs to Ω.

Picture

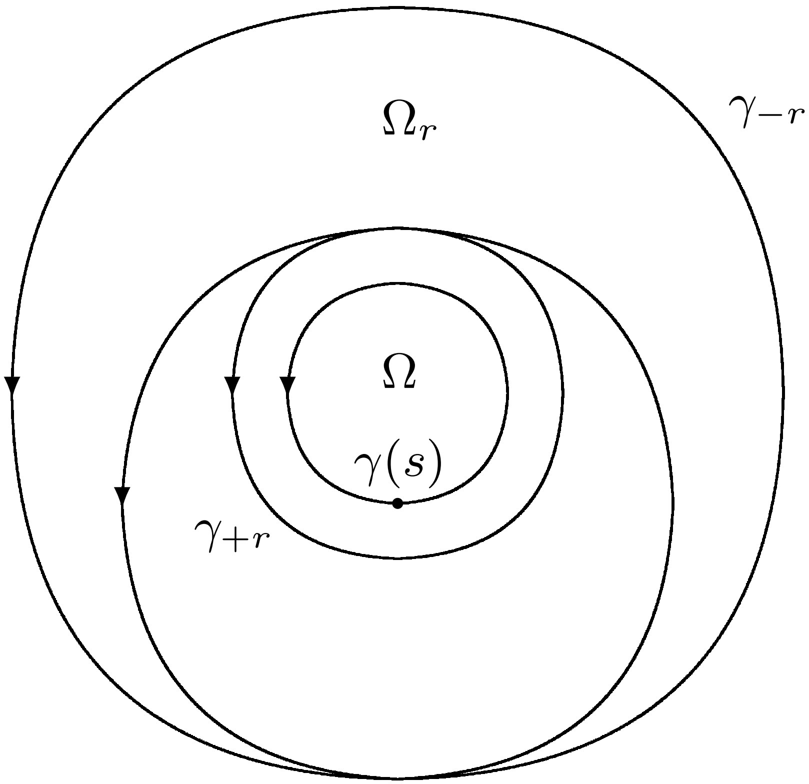

Fig. 2. Circle of radius r centered at $\gamma(s)$ is tangent to $\partial\Omega_r = \gamma_{+r} \cup \gamma_{-r}$.

Phase space Ω_r. Cylinder vs Annulus

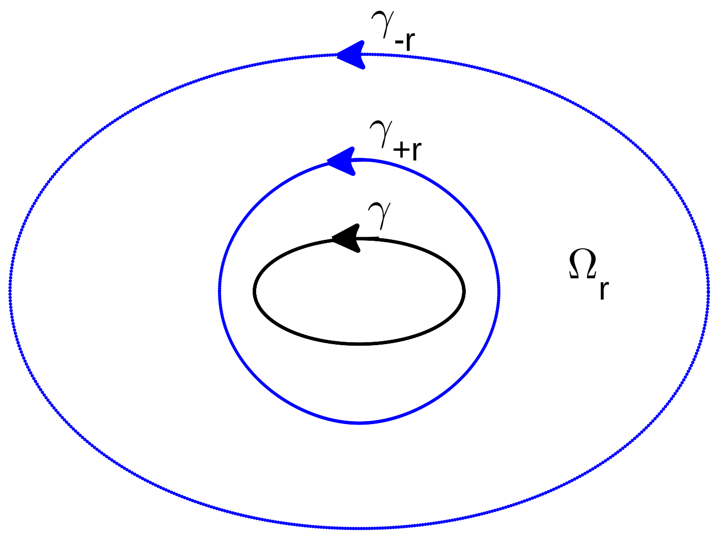

Figure 2: Phase cylinder $= \gamma \times (0, \pi)$; Phase annulus$= \Omega_r$.

Parallel curves

Moreover, for any smooth piece γ of the boundary $\partial\Omega$ we define two curves as follows. Fix an arc-length parameter s of positive (counterclockwise) orientation, we set:

(3) $$\gamma_{+r}(s) = \mathcal{L}(\gamma(s), \tau(s));\ \gamma_{-r}(s) = \mathcal{L}(\gamma(s), -\tau(s)),$$

where $\tau(s) = \dot{\gamma}(s)$. It is easy to see that, $\Omega_r \subset \mathbb{R}^2$ is a bounded domain in the plane homeomorphic to the annulus and the curves $\gamma_{\pm r}$, called parallel curves to γ, lie on the boundary $\partial\Omega_r$. Here γ_{-r} lies on the outer boundary of the annulus, and γ_{+r} lies on the inner boundary. The curves $\gamma_{\pm r}$ are also called equidistant curves, or fronts, in Singularity theory, or offset curves in Computer Aided Geometric Design.

Moreover, we introduce the mapping $\mathcal{M} : \Omega_r \to \Omega_r$ by the following rule: Let C_-, C_+ are two Larmor circles centered at P_-, P_+ respectively. We define

$$\mathcal{M}(P_-) = P_+ \iff C_- \quad \text{is transformed to} \quad C_+,$$

after billiard reflection at the boundary $\partial\Omega$. The map $\mathcal{M} : \Omega_r \to \Omega_r$ preserves the standard symplectic form in the plane, and thus Ω_r naturally becomes the phase space of magnetic billiard. We shall call $\mathcal{M}$ Magnetic billiard map. On the boundaries $\gamma_{\pm r}$, map $\mathcal{M}$ acts identically

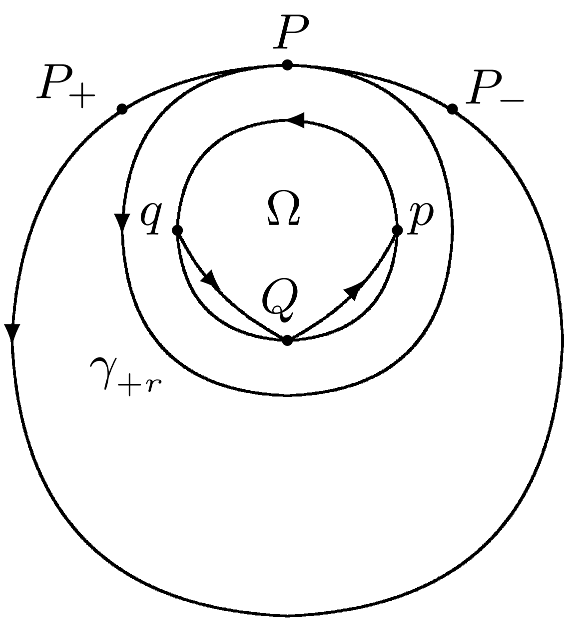

Fig. 3 Arcs $(q; Q)$ and $(Q; p)$ belong to C_- and C_+.

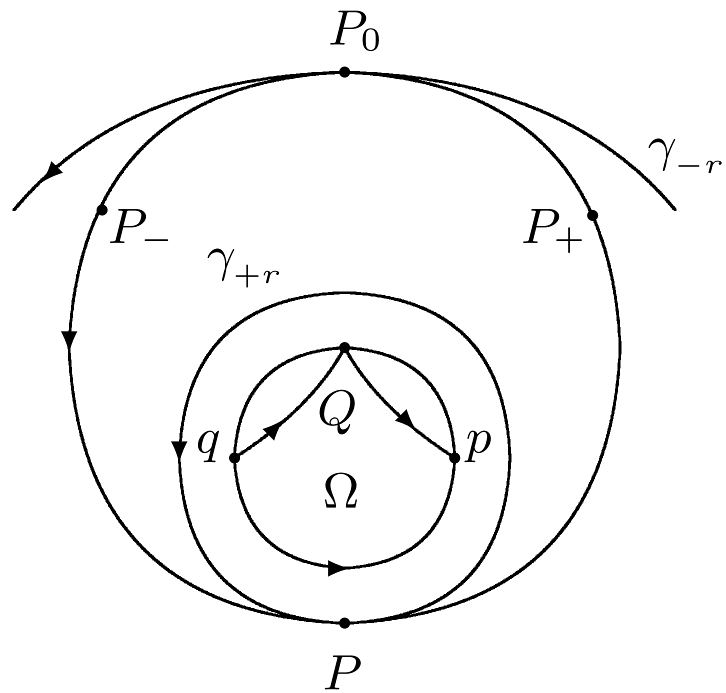

Fig. 4 Arcs $(q;Q)$ and $(Q;p)$ belong to C_- and C_+.

Symplectic form

Theorem The map $\mathcal{M}$ preserves the standard symplectic form ω in the plane.

Proof. $\mathcal{M}$ maps P_- to P_+. Let s be the arc-length parameter along γ. Let (τ, n), where $\tau = \dot{\gamma}, n = J\tau$, be the Frenet orthonormal frame along γ. Introduce the coordinates (s, ϵ) in Ω_r by

$$(s, \epsilon) \mapsto P = \gamma(s) + rJR_\epsilon \tau(s).$$

Notice that the points P_-, P_+ correspond the opposite signs of the coordinate ϵ. Differentiating with respect to s and ϵ we get

$$\frac{\partial P}{\partial s} = \tau(s) + krJR_\epsilon n(s) = \tau(s) - krR_\epsilon \tau(s),$$

$$\frac{\partial P}{\partial \epsilon} = rJR_\epsilon J\tau(s) = -rR_\epsilon \tau(s) = -r\cos\epsilon\, \tau(s) - r\sin\epsilon\, n(s).$$

$$\omega\left(\frac{\partial P}{\partial s}, \frac{\partial P}{\partial \epsilon}\right) = \omega\left(\tau(s), -rR_\epsilon \tau(s)\right) = -r\sin\epsilon.$$

$$\omega = -r\sin\epsilon\, ds \wedge d\epsilon.$$

Notice that the sign change of ϵ leaves this expression invariant, proving the invariance of ω under $\mathcal{M}$. Notice, ω coincides with the familiar invariant 2-form for the ordinary Birkhoff billiard.

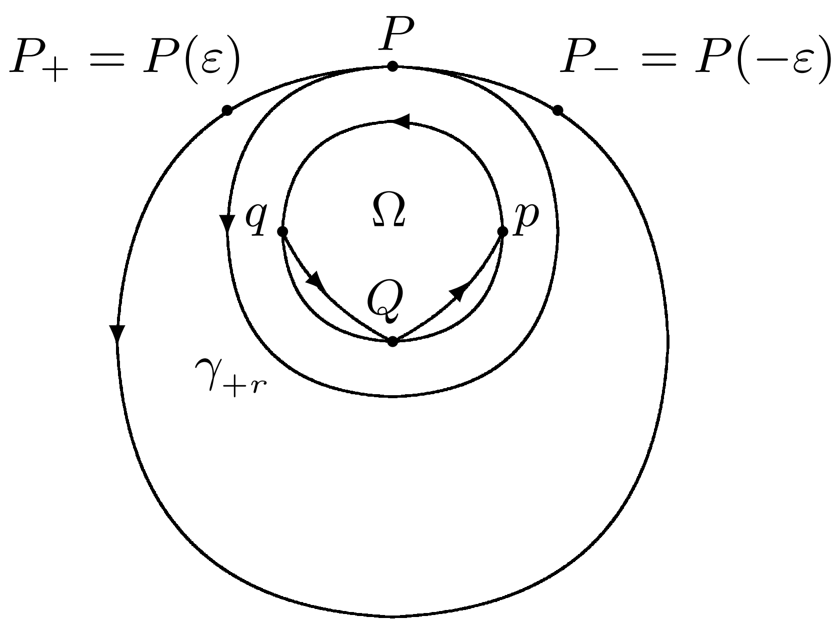

Fig. 5 Arcs $(q;Q)$ and $(Q;p)$ belong to C_- and C_+.

A theorem

Given a polynomial integral $\Phi = \sum_{k+l=0}^{N} a_{kl}(x) v_1^k v_2^l$ of the magnetic billiard we define:

(4) $$F : \Omega_r \to \mathbb{R}, \qquad F \circ \mathcal{L} = \Phi.$$

This is a well defined construction since Φ is an integral of the magnetic flow so has constant values on any Larmor circle. Moreover, since Φ is invariant under the billiard flow, it follows that F is invariant under billiard map $\mathcal{M}$:

$$F \circ \mathcal{M} = F.$$

Since Φ is a polynomial in v, then function F satisfies the following property: F restricted to any circle of radius r lying in Ω_r is a trigonometric polynomial of degree at most N.

Theorem. Let Ω_r be a domain in $\mathbb{R}^2$ which is the union of all circles of radius r whose centers run over a domain Ω (for example the whole $\mathbb{R}^2$). Let $F : \Omega_r \to \mathbb{R}$ be a continuous function on Ω_r such that F being restricted to any circle of radius r of Ω_r is a trigonometric polynomial of degree at most N. It then follows that F is a polynomial in (x, y) of degree at most $2N$.

Main result on magnetic billiard

Proposition. Suppose that the magnetic billiard in Ω admits a polynomial integral Φ and let F be the corresponding polynomial on $\overline{\Omega}_r$. Then for every smooth piece γ of the boundary $\partial\Omega$ it follows that

$$F|_{\gamma_{\pm r}} = const.$$

Corollary. The curves $\gamma, \gamma_{\pm r}$ are algebraic curves. Let $f_{\pm r}$ be a minimal defining polynomial of $\gamma_{\pm r}$. It may happen that both $\gamma_{\pm r}$ belong to the same component, so that $f_{+r} = f_{-r}$. For instance, if γ is ellipse, then $f_{-r} = f_{+r}$ is irreducible polynomial of degree 8.

Theorem. Let Ω be a convex bounded domain with a piece-wise smooth boundary, such that every smooth piece of the boundary has curvature at least β. Suppose that the magnetic billiard in Ω admits a Polynomial integral Φ. Then the following alternative holds: either $\partial\Omega$ is a circle, or every smooth piece γ of the boundary $\partial\Omega$ is not circular and has the property that affine curves $\{f_{\pm r} = 0\}$ are smooth in $\mathbb{C}^2$. Moreover, any non-singular point of intersection of the projective curve $\{\tilde{f}_{\pm r} = 0\}$ in $\mathbb{C}P^2$ with the infinite line $\{z = 0\}$ away from the isotropic points $(1 : \pm i : 0)$ must be a tangency point with the infinite line. Here $\tilde{f}_{\pm r}$ is a homogenization of $f_{\pm r}$.

Corollaries

Corollary1. For any non-circular domain Ω in the plane, the magnetic billiard inside Ω is not algebraically integrable for all but finitely many values of β.

Indeed, $f_{\pm r}$ is a polynomial in x, y and r. Every piece γ has positive curvature bounded from below by β, then there is an open interval $r \in (\rho_{\min}; \rho_{\max})$ where singularities of the parallel curve γ_{+r} are visible. So the equations

$$\partial_x f_{+r} = \partial_y f_{+r} = f_{+r} = 0$$

define an algebraic set in $\mathbb{C}^3(x, y, r)$ and its projection on $\mathbb{C}(r)$ is a constructible set containing the interval $r \in (\rho_{\min}; \rho_{\max})$. Thus this set is co-finite.

Corollary2. Let Ω be an interior of the standard ellipse:

$$\gamma = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}, \quad 0 < b < a.$$

Then for all $0 < \beta < k_{\min} = \frac{b}{a^2}$ magnetic billiard in ellipse is not algebraically integrable.

Parallel curves to ellipse

Parallel curves to ellipse are determined by:

$$a^8(b^4+(r^2-y^2)^2-2b^2(r^2+y^2))+b^4(r^2-x^2)^2(b^4-2b^2(r^2-x^2+y^2)+(x^2+y^2-r^2)^2)$$

$$-2a^6(b^6+(r^2-y^2)^2(r^2+x^2-y^2)-b^4(r^2-2x^2+3y^2)-b^2(r^4+3y^2(x^2-y^2)+$$

$$r^2(3x^2+2y^2)))+2a^2b^2(-b^6(r^2+x^2)-(-r^2+x^2+y^2)^2(r^4-x^2y^2-r^2(x^2+y^2))+$$

$$b^4(r^4-3x^4+3x^2y^2+r^2(2x^2+3y^2))+b^2(r^6-2x^6+x^4y^2-3x^2y^4+r^4(-4x^2+2y^2)+$$

$$r^2(5x^4-3x^2y^2-3y^4)))+a^4(b^8+2b^6(r^2+3x^2-2y^2)+(r^2-y^2)^2(-r^2+x^2+y^2)^2-$$

$$2b^4(3r^4-3x^4+5x^2y^2-3y^4+4r^2(x^2+y^2))+2b^2(r^6-3x^4y^2+x^2y^4-2y^6+$$

$$2r^4(x^2-2y^2)+r^2(-3x^4-3x^2y^2+5y^4)))=0.$$

It turns out to be irreducible. Moreover the parallel curves $\gamma_{\pm r}$ have singularities in the Complex plane for every $r > \frac{1}{k_{\min}} = \frac{a^2}{b} > b$ as follows:

$$(0, \pm\frac{\sqrt{b^2-a^2}\sqrt{a^2-r^2}}{a}),\ (\pm\frac{\sqrt{a^2-b^2}\sqrt{b^2-r^2}}{b}, 0).$$

Therefore the result follows from the main Theorem.

Remarkable equation

(5) $$F(P(-(\epsilon)) = F(P(+(\epsilon))$$

(6) $$F\left(x + r\frac{F_x(1-\cos\epsilon) + F_y \sin\epsilon}{|\nabla F|}; y + r\frac{F_y(1-\cos\epsilon) - F_x \sin\epsilon}{|\nabla F|}\right) -$$

$$F\left(x + r\frac{F_x(1-\cos\epsilon) - F_y \sin\epsilon}{|\nabla F|}; y + r\frac{F_y(1-\cos\epsilon) + F_x \sin\epsilon}{|\nabla F|}\right) = 0.$$

The next step is to expand equation (6) in power series in ϵ. The coefficient at ϵ^3 reads:

(7) $$(F_{xxx}F_y^3 - 3F_{xxy}F_y^2F_x + 3F_{xyy}F_yF_x^2 - F_{yyy}F_x^3) +$$

$$3\beta(F_x^2 + F_y^2)^{\frac{1}{2}}(F_{xx}F_xF_y + F_{xy}(F_y^2 - F_x^2) - F_{yy}F_xF_y) = 0, \quad (x, y) \in \gamma_{\pm r}.$$

Remarkably, the left-hand side of (7) is a complete derivative along the tangent vector field v to $\gamma_{\pm r}$, $v = (F_y, -F_x)$, of the following expression which therefore must be constant:

(8) $$H(F) + \beta|\nabla F|^3 = const, \quad (x, y) \in \gamma_{\pm r}, \text{ where}$$

$$H(F) := F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2.$$

Notice (8) is valid only for those points, where ∇F does not vanish. If the polynomial F is not a minimal polynomial defining γ_{+r} this can be false. So denote by f_{+r} irreducible defining polynomial of γ_{+r}, the proof for the curve γ_{-r} is completely the same. Then we have:

$$F = f_{+r}^k \cdot g,$$

for some integer $k \geq 1$, and polynomial g not vanishing on γ_{+r} identically. Given an arc of γ_{+r} where g does not vanish we may assume it is positive on the arc (otherwise we change the sign of F). Moreover, since f_{+r} is irreducible polynomial, then we may assume that ∇f_{+r} does not vanish on the arc. Therefore the equation (8) holds for $F^{\frac{1}{k}} = f_{+r} \cdot g^{\frac{1}{k}}$. Thus we have

(9) $$H(f_{+r} \cdot g^{\frac{1}{k}}) + \beta |\nabla (f_{+r} \cdot g^{\frac{1}{k}})|^3 = const, \quad (x,y) \in \{f_{+r} = 0\}.$$

Using the identities which are valid for all $(x,y) \in \{f_{+r} = 0\}$

$$H(f_{+r} \cdot g^{\frac{1}{k}}) = g^{\frac{3}{k}} H(f_{+r}), \quad \nabla (f_{+r} \cdot g^{\frac{1}{k}}) = g^{\frac{1}{k}} \nabla (f_{+r}),$$

we obtain from (9):

(10) $$g^{\frac{3}{k}} (H(f_{+r}) + \beta |\nabla f_{+r}|^3) = const, \quad (x,y) \in \gamma_{+r}.$$

Raising to the power k back we get:

(11) $$g^3(H(f_{+r}) + \beta|\nabla f_{+r}|^3)^k = const, \quad (x,y) \in \gamma_{+r}.$$

Proposition. The constant in equation (11) cannot be 0.

Proof. Recall important formulas for the curvature k of the curve defined implicitly by $\{f_{+r} = 0\}$:

(12) $$\text{div}\left(\frac{\nabla f_{+r}}{|\nabla f_{+r}|}\right) = \frac{H(f_{+r})}{|\nabla f_{+r}|^3} = \pm k_{+r}.$$

Now we take any point on γ_{+r} and substitute into (11). This gives that the constant must be non-zero. Indeed, if the $const$ is zero, then

$$\frac{H(f_{+r})}{|\nabla f_{+r}|^3} = -\beta.$$

Then by (12) we have $k_{+r} = \pm\beta$. But

$$\rho_{+r} = r - \rho \Rightarrow k_{+r} = \frac{k(\gamma)}{r \cdot k(\gamma) - 1} \Rightarrow k_{+r} > \frac{1}{r} = \beta.$$

End of proof

Consider now the equation (11) in $\mathbb{C}^2$. It follows from (11) and Proposition 26 that the curve $\{f_{+r}=0\}$ has no singular points in $\mathbb{C}^2$, since at singular points both $H(f_{+r})$ and $\nabla(f_{+r})$ vanish. Moreover, consider now in $\mathbb{C}P^2$ with homogeneous coordinates $(x:y:z)$ the projective curve $\{\tilde{f}_{+r}=0\}$. We shall denote homogeneous polynomials corresponding to f,g by $\tilde{f},\tilde{g}$ respectively. Then the homogeneous version of (11) for $(x:y:z)\in\{\tilde{f}_{+r}=0\}$ reads:

(13) $$\tilde{g}^3\left(z\cdot H(\tilde{f}_{+r})+\beta((\tilde{f}_{+r})_x^2+(\tilde{f}_{+r})_y^2)^{\frac{3}{2}}\right)^k=const\cdot z^p.$$

Here the power $p=3\deg g+3k(\deg f_{+r}-1)$ must be positive unless the degree of the polynomial f_{+r} and of F is one. But this is impossible, due to our convexity assumptions. Let Z be any point of intersection of $\{\tilde{f}_{+r}=0\}$ with infinite line $\{z=0\}$. Then by (13) for such a point we have two relations

$$(\tilde{f}_{+r})_x^2+(\tilde{f}_{+r})_y^2=0,\quad x(\tilde{f}_{+r})_x+y(\tilde{f}_{+r})_y+z(\tilde{f}_{+r})_z=x(\tilde{f}_{+r})_x+y(\tilde{f}_{+r})_y=0.$$

But these two relations are compatible only in the two cases: either

$$x^2+y^2=z=0,\quad or\quad (\tilde{f}_{+r})_x=(\tilde{f}_{+r})_y=0.$$

This completes the proof.

Outer magnetic billiard

Billiard map $\mathcal{M}$ coincides with, what we call *Outer magnetic billiard*. In addition, the result which we get by our method provides the extension of Theorem by Tabachnikov to the case of Outer magnetic billiards. Notice that there are two different cases:

1) In this case the orientation on Γ is clockwise then T is well defined for any $\beta > 0$.

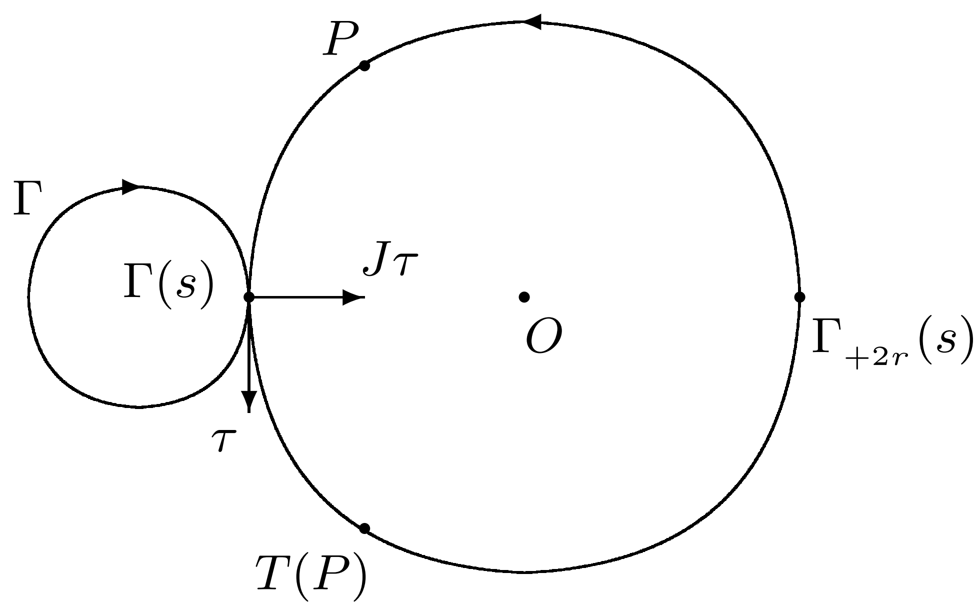

Fig. 6 Outer billiard map $P \to T(P)$ for clockwise orientation on Γ.

Outer magnetic billiards

2) However, if the orientation on Γ is counterclockwise then T is well defined for $0 < \beta < k_{min}$.

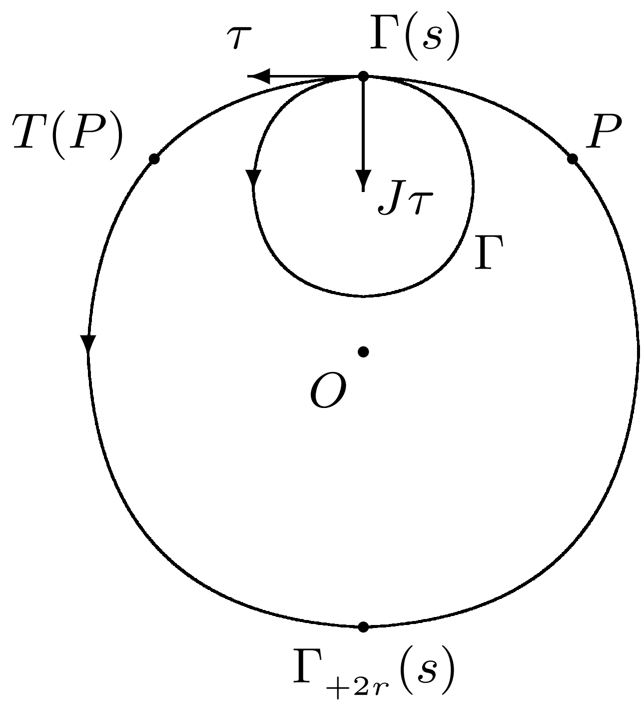

Fig. 7 Outer billiard map $P \to T(P)$ for counterclockwise orientation on Γ.

In both cases 1) and 2) the domain where the Outer billiard map T is defined is the annulus A bounded by Γ and Γ_{+2r}. The map $T : A \to A$ is a symplectic, we call T– Outer magnetic billiard. Next theorem shows in case 2) T and $\mathcal{M}$ are the same:

Inner and Outer magnetic billiards

Theorem. Magnetic billiard map $\mathcal{M} : \Omega_r \to \Omega_r$ coincides with Outer billiard T determined by the inner boundary γ_{+r} of Ω_r, endowed with a counterclockwise orientation.

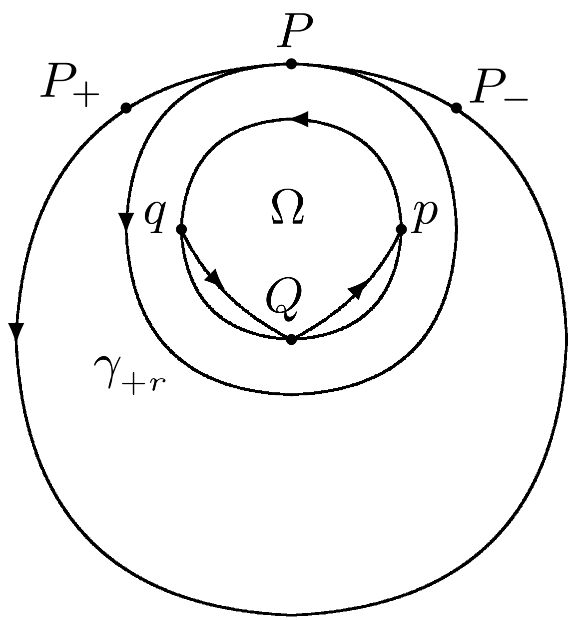

Fig. 8 Arcs $(q;Q)$ and $(Q;p)$ belong to C_- and C_+.

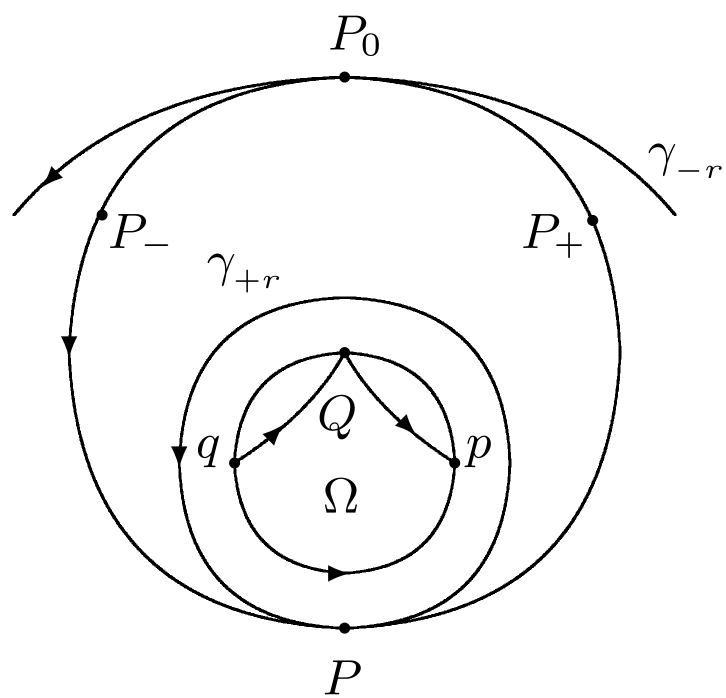

Fig. 9 Arcs $(q; Q)$ and $(Q; p)$ belong to C_- and C_+.

Remark. Notice that the outer billiard map T in the case 1) (see Figure 3) is not isomorphic to Magnetic Birkhoff billiard globally, by topological reasons. Indeed the difference between rotation numbers of two boundaries Γ, Γ_{+2r} equals 0 for case 1) and equals 2π in case 2). Nevertheless, since our method below is concentrated near the boundaries it applies for both cases 1) and 2).

Theorem for Outer magnetic billiards

Theorem. Assume that there exists a non-constant Polynomial F such that F is invariant under Outer billiard map T. Let f, f_{+2r} are irreducible defining polynomials of Γ, Γ_{+2r}. Then the following alternative holds. Either Γ is a circle, or the curves $\{f = 0\}, \{f_{+2r} = 0\}$ in $\mathbb{C}^2$ are smooth with the property that any non-singular intersection point of the projective curves $\{\tilde{f} = 0\}, \{\tilde{f}_{+2r} = 0\}$ in $\mathbb{C}P^2$ (here $\tilde{f}$ is a homogenization of f) with the infinite line $\{z = 0\}$ which is not an isotropic point $(1 : \pm i : 0)$, must be a point of tangency.

Corollary 1 The outer magnetic billiard for ellipse is not algebraically integrable.
Proof. For the case of ellipse the curve $\{\tilde{f} = 0\}$ is smooth everywhere in $\mathbb{C}P^2$ and intersects transversally the infinite line in two points away from the isotropic points.

Corollary 2 For all but finitely many values of the magnitude of magnetic field β, the Outer magnetic billiard of Γ is not algebraically integrable unless Γ is a circle.

Proof of the lemma

Lemma. Let F be a C^∞ function $F : A \to \mathbb{R}$ where

$$A = \{(x, y) : (r-\delta)^2 \le x^2 + y^2 \le (r+\delta)^2\},$$

is the annulus in $\mathbb{R}^2$. Suppose that the function F being restricted to any circle of radius r lying in A is a trigonometric polynomial of degree at most N. It then follows that F is a polynomial in x and y of degree at most $2N$.

Proof. (based on an idea of S. Tabachnikov). We shall say that F has property P_N if the restriction of F to any circle of radius r lying in A is a trigonometric polynomial of degree at most N. The proof of Lemma goes by induction on the degree N.

1) For $N = 0$, Lemma obviously holds since if F has property P_0 then F is a constant on any circle of radius r and hence must be a constant on the whole A, because any two points of A can be connected by a union of finite number circular arcs of radius r.

2) Assume now that any function satisfying property P_{N-1} is a polynomial of degree at most $2(N-1)$.

Let F be any smooth function on A of property P_N. Denote by C_0 be the core circle of A, i.e. $C_0 = \{x^2 + y^2 = r^2\}$, and let F_0 be the polynomial in (x, y) of degree N satisfying $F|_{C_0} = F_0|_{C_0}$.

Then, one can find a C^∞ function $G : A \to \mathbb{R}$ so that

(14) $$F(x,y) - F_0(x,y) = (x^2 + y^2 - r^2)G(x,y), \quad \forall (x,y) \in A.$$

Let us show now that G has property P_{N-1}. Then by induction we will have that G is a polynomial of degree $2(N-1)$ and thus by (14), F is a polynomial of degree $2N$ at most. We need to show that the function $g := G|_C$ is a trigonometric polynomial of degree $(N-1)$ or less, for any circle C of radius r in A . With no loss of generality we may assume that the circle C is centered on the $x-$axes (otherwise apply suitable rotation of the plane). Then

$$C = \{(x,y) \in A : (x-a)^2 + y^2 = r^2\}, \quad |a| < \delta.$$

Substituting $x = a + r\cos t, y = r \sin t$ into (14) we have

$$(F - F_0)|_C = (a^2 + 2ar\cos t) \cdot g.$$

$$\sum_{-\infty}^{+\infty} f_k e^{ikt} = a(a + re^{it} + re^{-it}) \sum_{-\infty}^{+\infty} g_k e^{ikt},$$

where f_k are Fourier coefficients of $(F - F_0)|_C$.

Moreover, we have:

$$f_k = 0, \quad |k| > N,$$

since both F, F_0 have property P_N. And hence:

$$rg_{k+1} + ag_k + rg_{k-1} = 0, \quad |k| > N.$$

The characteristic polynomial of this difference equation

$$\lambda^2 + \frac{a}{r}\lambda + 1 = 0$$

has two complex conjugate roots $\lambda_{1,2} = e^{\pm i\alpha}$ and therefore we get the formula:

$$g_{N+l} = c_1 e^{il\alpha} + c_2 e^{-il\alpha}, \quad l \geq 2, \quad \text{where}$$

$$c_1 + c_2 = g_N, \quad c_1 e^{i\alpha} + c_2 e^{-i\alpha} = g_{N+1}.$$

It is obvious now that if at least one of the coefficients g_N or g_{N+1} does not vanish, then the sequence $\{g_{N+l}\}$ does not converge to 0 when $l \to +\infty$. This contradicts the continuity of g. Therefore both g_N, g_{N+1} must vanish and so g is a trigonometric polynomial of degree at most $(N-1)$, proving that G has property P_{N-1}. This completes the proof.

Proof of the Theorem.

Next we give the proof of the Theorem.

Proof. Take any circle of radius r lying in Ω_r and let A be the annulus which is the closure of its $\delta-$neighborhood. Using the convolution with a C^∞ mollifier ρ_ϵ compactly supported in a small disc of radius ϵ, we get a C^∞ function F_ϵ :

$$F_\epsilon(z) := \int \rho_\epsilon(z-\xi)F(\xi)d\xi = \int F(z-\xi)\rho_\epsilon(\xi)d\xi, \quad z=(x,y).$$

It is easy to see, that if F has property P_N then also F_ϵ has property P_N on the chosen annulus A for all ϵ small enough, $0<\epsilon<\epsilon_0$. Then by Lemma, F_ϵ must be a polynomial on A of degree at most $2N$, for $0<\epsilon<\epsilon_0$. Recall, that F_ϵ converge to F uniformly on A as $\epsilon \to 0$. Therefore, since the space of Polynomials of degree at most $2N$ is finite-dimensional it then follows that F is also a polynomial on A of degree at most $2N$. The set Ω_r can be covered by annuli like A, therefore F must be a polynomial of degree at most $2N$ on the whole Ω_r. This completes the of Theorem.

Questions

1. Notice our results hold for a fixed β. For all but finitely many β we have algebraic non-integrability. However there is still a hope for new integrable magnetic billiard, for some β. The reason for the optimism comes from recently discovered new integrable magnetic geodesic flows on the 2-torus: we proved that one can find a Riemannian metric on $\mathbb{T}^2$ and exact magnetic field β such that the magnetic flow has an additional quadratic integral on one energy level.

2. Question on magnetic flows which are integrable for all energy levels also seems to be not simple. Given a Riemannian metric on the 2-torus, and an exact magnetic field. Suppose there exists a polynomial in momenta integral of the magnetic flow of all energy levels. Does this imply that the system has an $\mathbb{S}^1$ symmetry?

3. Other magnetic toys:
Andreev billiard;

THANKS!

AND WELCOME TO THE PLAYGROUND!