Right-angled hyperbolic polyhedra, hyperbolic links and Vol-Det conjecture

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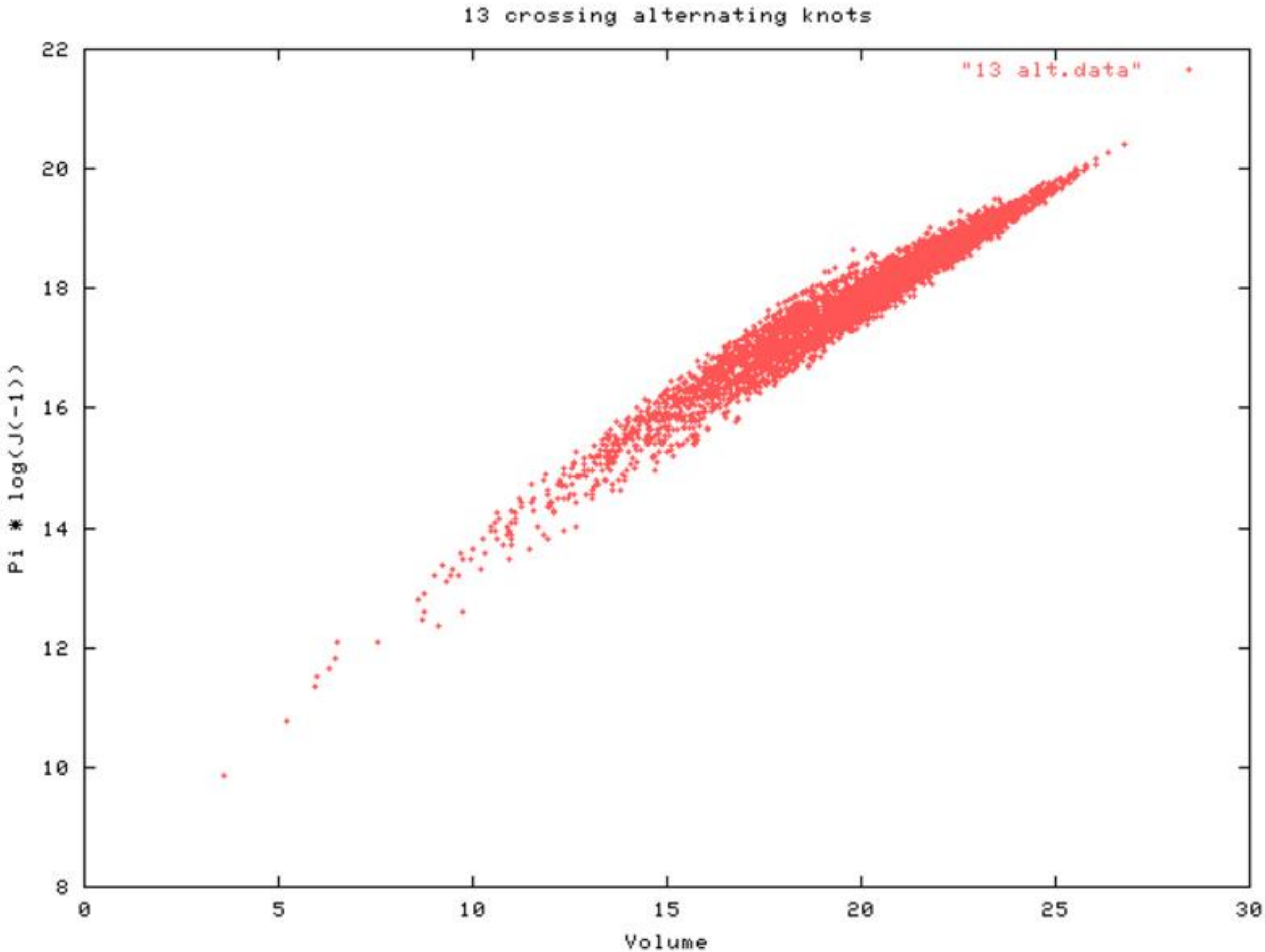


13 crossing alternating knots
"13 alt.data"
Pi * log(J(-1))
Volume
22
20
18
16
14
12
10
8
0
5
10
15
20
25
30

Determinant of the link

For a link K define determinant as the following integer

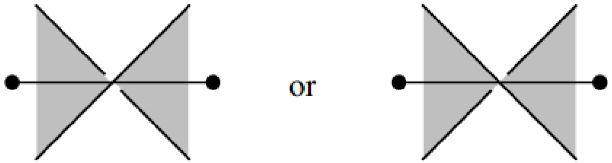
$$\det(K) = |\Delta_K(-1)| = |V_K(-1)|$$

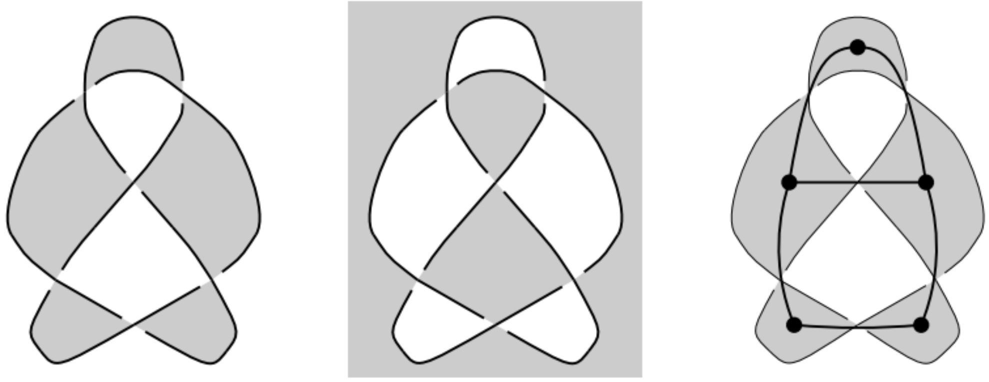
Example. Determinants of the initial list of hyperbolic knots.

Knot K	4_1	5_2	6_1	6_2	6_3	7_2	7_3	7_4	7_5	7_6	7_7
det(K)	5	7	9	11	13	11	13	15	17	19	21

Checkerboard graph

A knot or link diagram is *alternating* if the crossings alternate under, over, under, over, as one travels along each component of the link. A link is *alternating* if it has an alternating diagram.
Consider alternating link diagram D and associate to it its checkerboard graph $\boldsymbol{\Gamma}(D)$.





Left and middle: Two checkerboard colorings. **Right**: checkerboard graph for the first coloring.

Determinant and spanning trees

Theorem (Crowell, 1959) If D is an alternating diagram of a link K, then

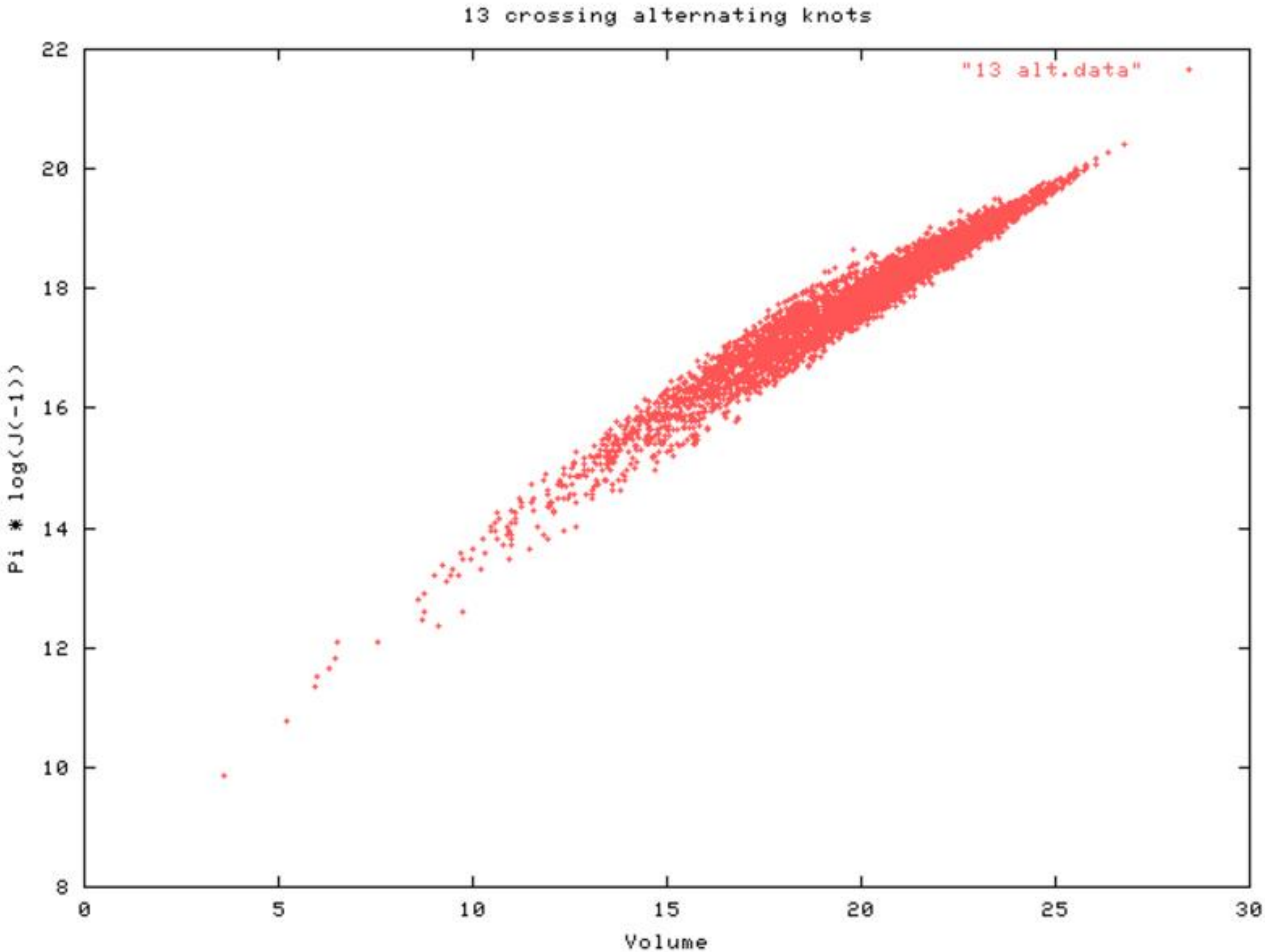
$$\det(K) = \#\ of\ spanning\ trees\ of\ \boldsymbol{\Gamma}(D).$$

Volume of the link complement

A link L is called hyperbolic if a complete Riemannian metric of constant negative curvature (a hyperbolic structure) can be introduced on $S^3\setminus\mathrm{L}$.

By Mostow's theorem, such a hyperbolic structure is unique. So $vol(S^3\setminus\mathrm{L})$ is an invariant of the link.

Hoste, Thistlethwaite, Weeks [1991]: Among the 1,701,903 knots with at most 16 crossings, only 32 are not hyperbolic



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Formalization

Dunfield's Observation [2000]:
$vol(K) \approx a \log \det(K) + b.$

Impossible to replace $\approx$ with $\geq$.

But we can try to replace $\approx$ with $\leq$.

Vol-Det conjecture

Conjecture (Champanerkar, Kofman, Purcell, 2015) Let K be an alternating hyperbolic link. Then

$$vol(K) < 2\pi \log \det(K)\,.$$

A weaker estimate was previously proven by Stoimenow.

Theorem (Stoimenow, 2007) Let K be an alternating hyperbolic link. Then

$$vol(K) \leq 28{,}639760 \cdot \log \det(K) \; - 15{,}791802.$$

Proof scheme of the last estimate

$$vol(K) \leq \det(K)$$

$\Downarrow$

$vol(K)$ $\leq$ Combinatorial parameters of the diagram $\leq$ $\det(K)$

Volume bounds for complements of hyperbolic links

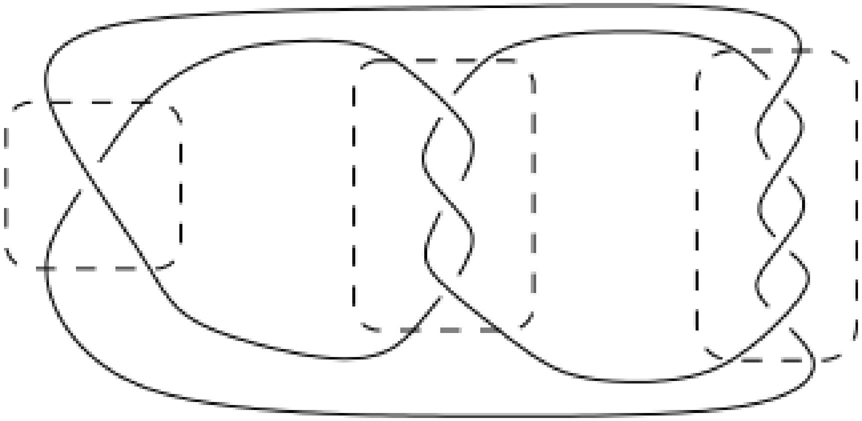
Adams' volume bound for knots

Theorem (C.Adams, 1983)
Let D be a diagram of a hyperbolic knot K with $c(D)$ crossings. If K is not a figure-eight knot, then

$$vol(S^3 \backslash K) \leq v_{tet} \cdot (4c(D) - 16).$$

Twists

The diagram below has 3 twists.
Denote the number of twists in the diagram D by t(D).



Volume bound in terms of number of twists

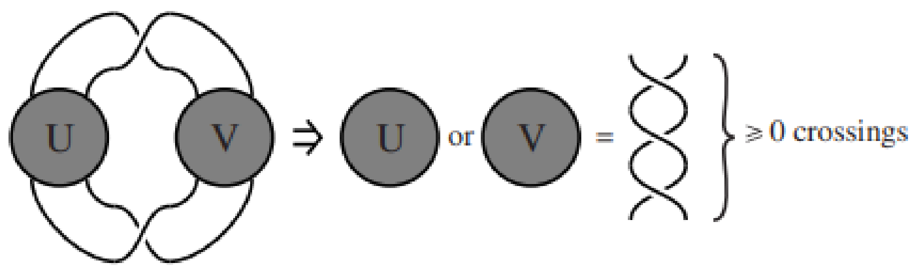
Theorem (I.Agol, D.Thurston, 2004)
Let D be a diagram of a hyperbolic link K with $\mathrm{t}(D)$ twists. Then

$$vol(S^3 \setminus K) \leq 10 \cdot v_{tet} \cdot (t(D) - 1)$$

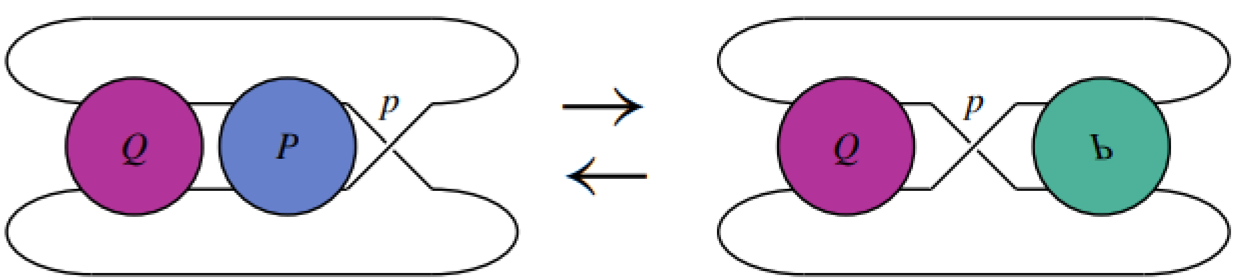
M. Lackenby, *The volume of hyperbolic alternating link complements*. With an appendix by I. Agol and D. Thurston, Proceedings of the London Mathematical Society 88 (2004), 204–224.

Twist-reduced diagrams

The diagram of a link is *twist-reduced* if whenever a simple closed curve in the diagram intersects the link projection transversely in four points disjoint from the crossings, and two of these points are adjacent to some crossing, and the remaining two points are adjacent to some other crossing, then this curve bounds a subdiagram that consists of a (possibly empty) collection of bigons arranged in a row between these two crossings.



twist-reduced diagram

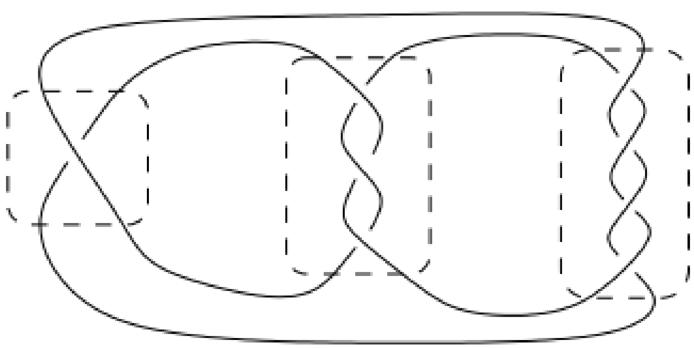


Flype

Taking into account sizes of the twists

The bound can be improved if we take into account not only the number of twists, but also the size of the twists (Tsvetkova-Dasbach, C.Adams).

But only twists of length less than 5 have a smaller contribution to the volume estimate.



New volume bound

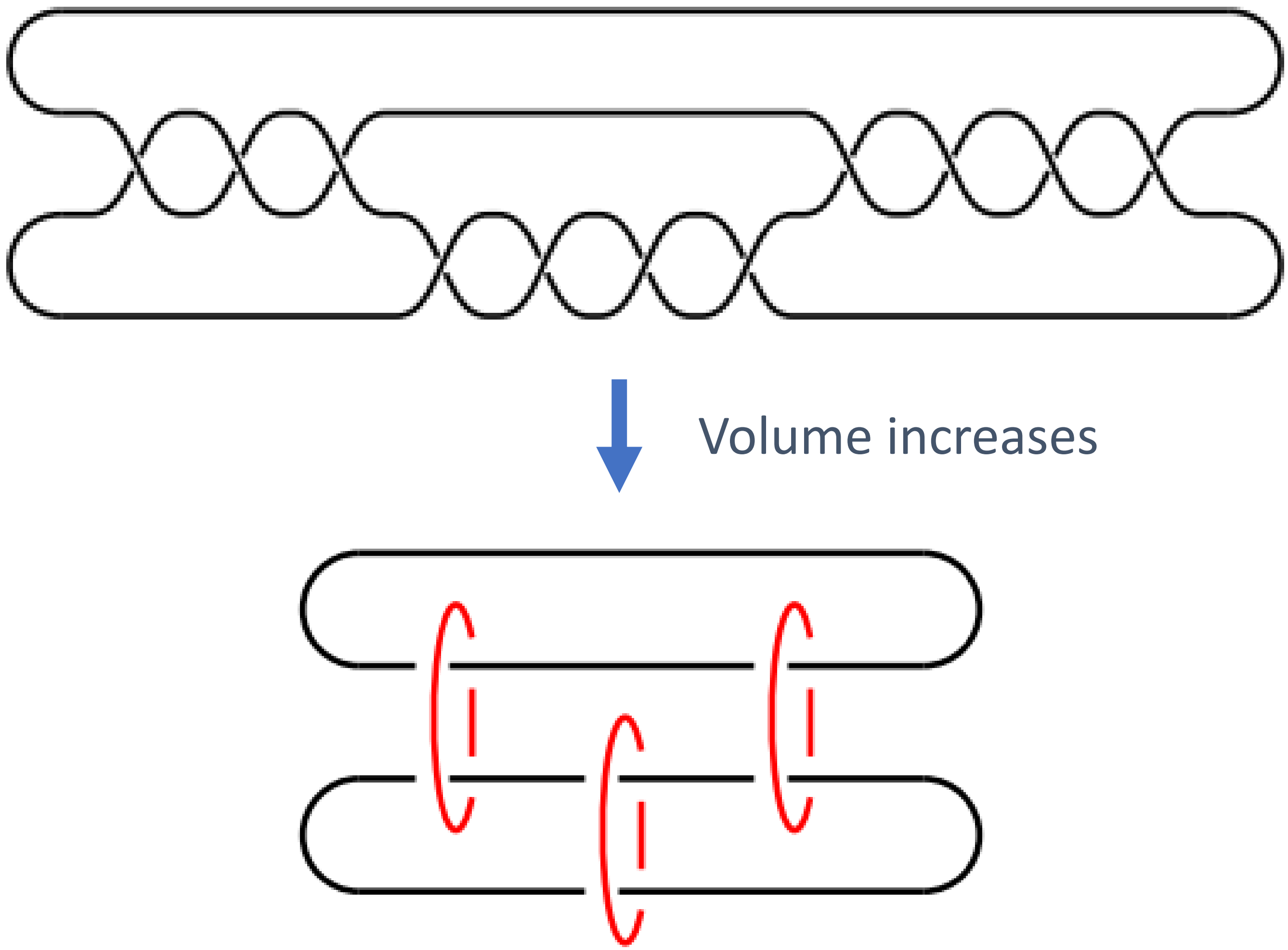
Theorem (Vesnin, E., 2024)

Let D be a diagram of a hyperbolic link K with $t(D) > 8$ twists. Then

$$vol(S^3 \setminus K) \leq 10 \cdot v_{tet} \cdot (t(D) - 1.4)$$

А.Ю. Веснин, А. А. Егоров, *Верхние оценки объемов обобщенных гиперболических многогранников и зацеплений*, Сибирский математический журнал **65:3** (2024), 534-551.

Proof: augmentation



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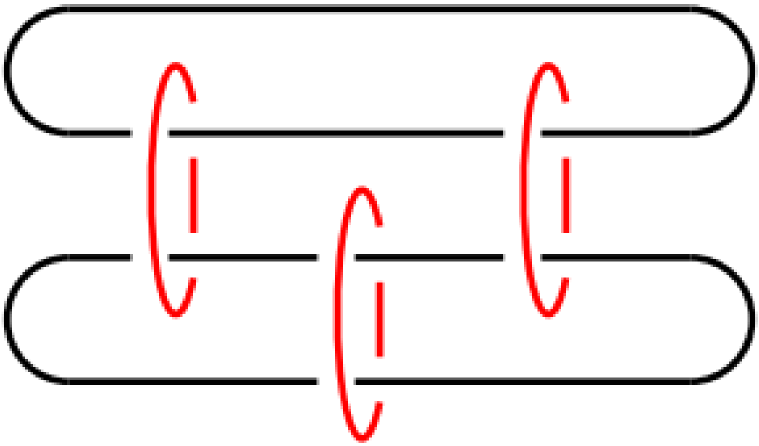
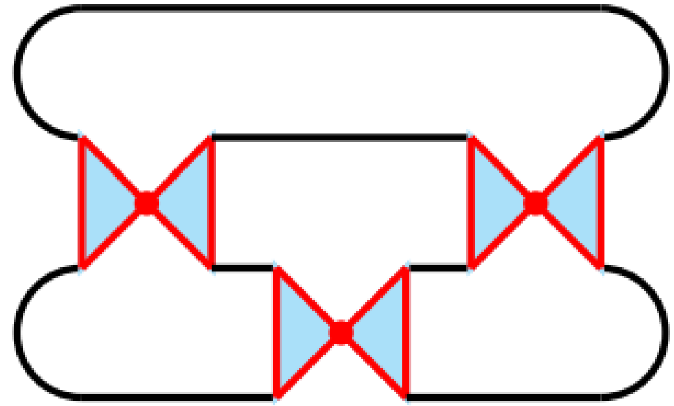
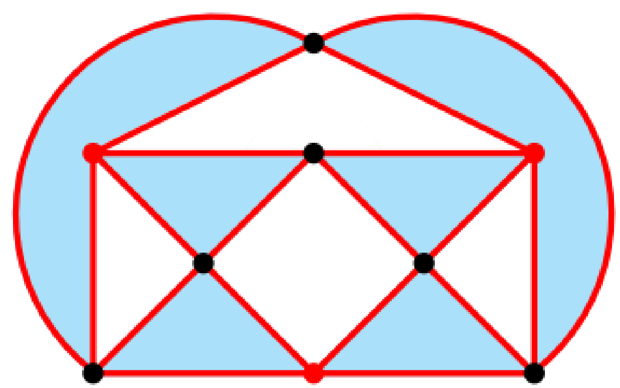
Proof: from link compliment to polyhedra







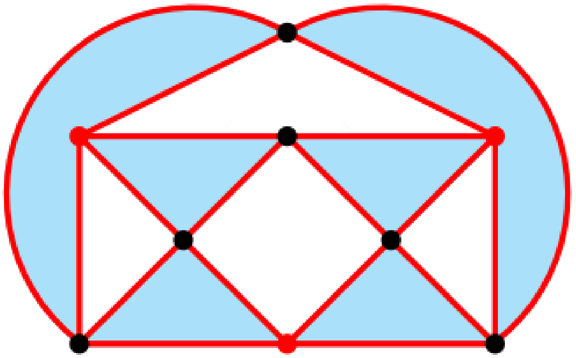
The complement to the resulting link K_a can be represented as the union of two identical ideal right-angled polyhedra P_1 и P_2. And then $vol(K_a)=vol(P_1)+vol(P_2)$.

Proof: ideal polyhedra

Now we have to estimate volumes of these two *ideal right-angled polyhedra*.

Theorem (Andreev, 1970) A polyhedral graph is realized as a graph of an ideal right-angled polyhedron in H^3 if and only if it is a cyclically *6-connected* graph *4-valent graph*. Moreover, such a realization is unique.

What is special about these type of ideal right-angled polyhedra?
One thing is - *they have a lot of triangular faces.*



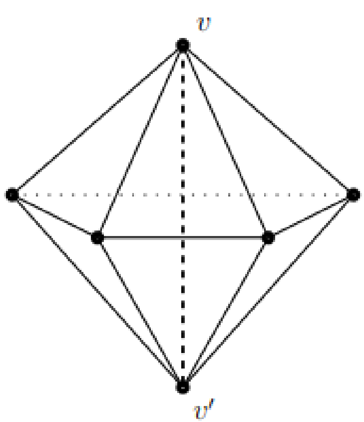
Proof: bounds for volume of polyhedra

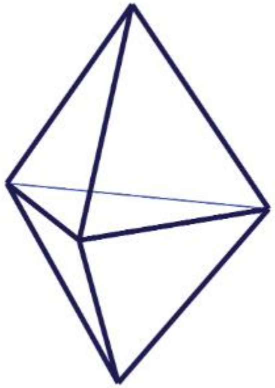
Lemma (E., Vesnin, 2024) Let P be an ideal hyperbolic right-angled polyhedron with V vertices and p_3 triangular faces. Then

$$vol(P) < 2v_{tet} \cdot (V - \frac{p_3 + 8}{4})$$

The proof of the lemma is as follows.

1. We choose a vertex of the polyhedron, connect it with the other vertices. We get a partition of the polyhedron into pyramids.
2. We double the pyramids.
3. We estimate the volumes of the bipyramids.



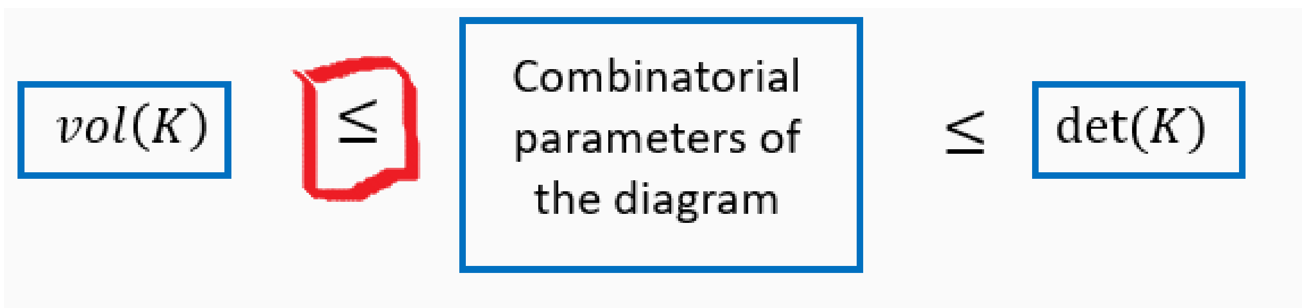


Let's return to volume estimates through determinant

Improvement of the Stoimenow bound for the case $t(D) > 8$.

Theorem (Stoimenow, 2007) Let K be an alternating hyperbolic link. Then

$$vol(K) \leq 28{,}639760 \cdot \log \det(K) \; - 15{,}791802.$$



Theorem (Vesnin, E., 2024+) Let K be an alternating hyperbolic link and D is its reduced alternating twist-reduced diagram with $t(D) > 8$. Then

$$vol(K) \leq 28{,}639760 \cdot \log \det(K) \; - 19{,}851568.$$

Vol-Det conjecture

Conjecture (Champanerkar, Kofman, Purcell, 2015) Let K be an alternating hyperbolic link. Then

$$vol(K) < 2\pi \log \det(K) .$$

Champanerkar, Kofman, and Purcell showed that the conjecture is true:

- For all alternating knots with crossing number no more than 16.
- For several infinite families of knots of a special type.

The constant 2π is sharp; specifically, for any $\alpha < 2\pi$, there exists an alternating knot K_α such that

$$\alpha \log \det(K_\alpha) < vol(K).$$

Barton [2019]: The conjecture is true for 2-bridge knots and knots that are closures of 3-braids.

Vol-Det conjecture

Theorem (Burton, 2018) Let K be an alternating hyperbolic link and its reduced alternating twist-reduced diagram D has t twists and c crossings. If

$$c > t + \xi^{t-1} - 2\gamma^{t-1},$$

where $\gamma \approx 1.4253$ and $\xi = e^{5v_3/4} \approx 5.0296$, then Vol-Det conjecture is true for K.

Theorem (E. , Vesnin, 2024+) In the conditions of the previous theorem if $t > 8$ and

$$c > t + \frac{\xi^{t-1.4}}{2\gamma^{\frac{t-3}{2}}} - 2\gamma^{\frac{t+1}{2}},$$

then Vol-Det conjecture is true for K.

Example: $t = 9$.

Let's consider the case $t = 9$ as an example. Barton's estimate says that the Vol-Det conjecture is true for links with the number of crossings $c > 409453$.

And the last theorem says that the Vol-Det conjecture is true for links with the number of crossings $c > 37055$.

Thanks for your attention!