Geometric hyperelliptic manifolds and Hamiltonian subcomplexes in right-angled polytopes

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Real moment-angle manifolds

- Toric topology assigns to each simple convex n-polytope P with m facets $F_1, \ldots, F_m$ a smooth orientable n-dimensional real moment-angle manifold

$$\mathbb{R}\mathcal{Z}_P = P^n \times \mathbb{Z}_2^m / \sim,$$

where $(p, a) \sim (q, b)$ iff $p = q$ and $p - q \in \langle e_i : p \in F_i \rangle$. Here $e_1, \ldots, e_m$ is the standard basis in $\mathbb{Z}_2^m$.

- The group $\mathbb{Z}_2^m$ acts on $\mathbb{R}\mathcal{Z}_P$ and $\mathbb{R}\mathcal{Z}_P / T^m \simeq P^n$.
- The equivariant topology of $\mathbb{R}\mathcal{Z}_P$ is determined by the combinatorics of P.

Manifolds defined by vector-colorings of polytopes

- Any subgroup $H(\Lambda) \subset \mathbb{Z}_2^m$ of rank $m - r$ is defined by a system of linear equations with an $r \times m$ matrix Λ.
- We call the mapping $F_i \to \Lambda_i$ a vector-coloring of rank r.
- The space

$$N(P, \Lambda) = \mathbb{R}\mathcal{Z}_P / H(\Lambda) = P^n \times \mathbb{Z}_2^r / \sim,$$

where $(p, a) \sim (q, b)$ iff $p = q$ and $a - b \in \langle \Lambda_i : p \in F_i \rangle$ is a pseudomanifold with an action of $\mathbb{Z}_2^r \simeq \mathbb{Z}_2^m / H$.

Free actions

- The action of $H(\Lambda)$ is free if and only if for any vertex $v = F_{i_1} \cap \cdots \cap F_{i_n}$ the images $\Lambda(F_{i_1}), \ldots, \Lambda(F_{i_n})$ are linearly independent.
- We call such vector-colorings linearly independent.
- In this case $N(P, \Lambda)$ is a smooth manifold.
- For $r = n$ and a free action the manifold $N(P, \Lambda)$ is called a small cover.
- The action of $\mathbb{Z}_2^n$ on a small cover is locally modelled by the action of $\mathbb{Z}_2^n$ on $\mathbb{R}^n$ by changes of signs.
- Small covers were introduced by M. Davis and T. Januszkiewicz in 1991.

Geometric manifolds

- The following construction was invented in the works by A.Yu. Vesnin and A.D. Mednykh (starting from 1985).
- Given a compact right-angled polytope P in some geometry X, where $X = \mathbb{S}^n$, $\mathbb{R}^n$, $\mathbb{L}^n$, or a product of such spaces, consider the right-angled Coxeter group

$$G(P) = \langle \rho_1, \dots, \rho_m | \rho_i^2 = 1, \rho_i\rho_j = \rho_j\rho_i \text{ if } F_i \cap F_j \neq \varnothing \rangle$$

  It is isomorphic to a subgroup of isometries of X generated by reflexions ρ_i in hyperplanes corresponding to facets F_i.
- An epimorphism $\varphi_\Lambda \colon G(P) \to \mathbb{Z}_2^r$, $\rho_i \to \Lambda_i$ such that for any vertex $v = F_{i_1} \cap \dots \cap F_{i_n} \neq \varnothing$ the images $\Lambda_{i_1}, \dots, \Lambda_{i_n}$ are linearly independent, gives a subgroup $G(P, \Lambda) = \mathrm{Ker}\varphi_\Lambda$ which acts freely on X.

$X/G(P, \Lambda)$ is a geometric manifold homeomorphic to $N(P, \Lambda)$.

Hyperelliptic manifolds

Definition

- A hyperelliptic manifold M^{n} is a topological manifold M^{n} with an involution τ such that the orbit space $\mathrm{M}^{\mathrm{n}}/\langle\tau\rangle$ is homeomorphic to S^{n}.
- the involution τ is called hyperelliptic.

Using a Hamiltonian cycle, a Hamiltonian theta-graph or a Hamiltonian K_4-graph on a right-angled polytope in $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{L}^3$, $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{L}^2 \times \mathbb{R}$ A.D. Mednykh and A.Yu.Vesnin have built examples of geometric hyperelliptic 3-manifolds.

Main result of the talk

We generalize their construction to the n-dimensional case.

When $N(P, \Lambda)$ is a manifold?

The following result was obtained jointly with Dmitry Gugnin and can be extracted from the general Mikhailova-Lange results.

Proposition (D. Gugnin-E., 24)

- $N(P, \Lambda)$ is a topological manifold if and only if for any face $F_{i_1} \cap \cdots \cap F_{i_k} \neq \varnothing$ different nonzero vectors among $\Lambda_{i_1}, \ldots, \Lambda_{i_k}$ are linearly independent.
- $\partial N(P, \Lambda)$ is glued of copies of facets F_i with $\Lambda_i = 0$.

The following result refines similar results by A.Yu. Vesnin, H. Nakayama – Y. Nishimura, and A. Kolpakov – B. Martelli – S. Tschantz

Proposition

A pseusomanifold $N(P, \Lambda)$ is closed and orientable if and only if in some basis $\Lambda_i = (1, \lambda_i)$ for all i.

Complex $\mathcal{C}(P, c)$

- A coloring c of P in r colors is a surjective mapping $c\colon \{F_1, \dots, F_m\} \to \{1, \dots, r\}$.
- It defines a complex $\mathcal{C}(P, c)$ with facets G_j the connected components of unions $\bigcup_{c(F_i)=\text{const}} F_i$ corresponding to the same color and faces the connected components of intersections of facets G_j.
- The complexes $\mathcal{C}(P, c_P)$ and $\mathcal{C}(Q, c_Q)$ are equivalent if there is a homeomorphism $P \to Q$ mapping bijectively facets of the first complex to facets of the second.

Proposition (E., 24)

In dimension $n = 3$ the complexes $\mathcal{C}(P, c)$ correspond to disjoint unions of simple closed curves and connected 3-valent bridgeless graphs on S^2.

Complexes $\mathcal{C}(n,k)$

Proposition (E., 24)

For the simplex Δ^n and any its coloring c in k colors the complex $\mathcal{C}(\Delta^n, c)$ is equivalent to the complex

$$S^n_k = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \dots + x_n^2 = 1, x_1 \geqslant 0, \dots, x_k \geqslant 0\}$$

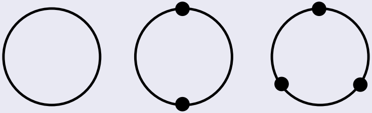
with facets $S^n_k \cap \{x_i = 0\}$, $i = 1, \dots, k$, and to the complex

$$B^n_k = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + \dots + x_n^2 \leqslant 1, x_1 \geqslant 0, \dots, x_{k-1} \geqslant 0\}$$

with facets $B^n_k \cap \{x_1^2 + \dots + x_n^2 = 1\}$ and $B^n_k \cap \{x_i = 0\}$, $i = 1, \dots, k-1$.

Examples

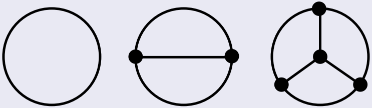
$n = 2$



C(2,1) C(2,2) C(2,3)

$n = 3$

- $\mathcal{C}(3,1)$ has one facet $\partial P \simeq S^2$;
- for $k > 1$ the projections of $C(3,k) \subset S^2$ to $\mathbb{R}^2$ look like



C(3,2) C(3,3) C(3,4)

$\mathcal{C}(n,k)$-subcomplexes and spheres $N(P,\Lambda)$

- By a $\mathcal{C}(n,k)$-subcomplex $C \subset \partial P$ we call a complex $C = \mathcal{C}(P,c)$ equivalent to $\mathcal{C}(n,k)$.
- A $\mathcal{C}(n,k)$-subcomplex $C \subset \partial P$ induces a vector-coloring Λ_C of rank k by the rule $F_i \to e_j$, where $e_1, \dots, e_k$ is some basis in $\mathbb{Z}_2^k$ and F_i lies in j-th facet of C. Λ_C is defined uniquely up to a change of coordinates in $\mathbb{Z}_2^k$.
- The space $N(P,\Lambda_C)$ is homeomorphic to S^n.

Theorem (E.,24)

If $n \leqslant 4$, the space $N(P,\Lambda)$ is homeomorphic to S^n if and only if $\Lambda = \Lambda_C$ for some $\mathcal{C}(n,k)$-subcomplex C.

Question

For $n \geqslant 5$ are the examples of $N(P,\Lambda) \simeq S^n$ not of this kind?

Partial results for general n

Theorem (E.,24)

Let $N(P,\Lambda) \simeq S^n$ and $\{G_1, \ldots, G_M\}$ be the set of facets of $\mathcal{C}(P,\Lambda)$. Then for each $\omega = \omega_1 \sqcup \cdots \sqcup \omega_k \subset [M]$ the intersection $\left(\bigcup_{i_1 \in \omega_1} G_{i_1}\right) \cap \cdots \cap \left(\bigcup_{i_k \in \omega_k} G_{i_k}\right)$ is

- a rational homology $(n-k)$-disk if $\omega \neq [M]$;
- a rational homology $(n-k)$-sphere if $\omega = [M]$.

In particular, $M \leqslant n+1$ and each k-face of the complex $\mathcal{C}(P,\Lambda)$ is either a k-RHD or a k-RHS.

Example

For any face $G = F_{i_1} \cap \dots \cap F_{i_k}$ and the standard basis $e_1, \dots, e_{k+1} \in \mathbb{Z}_2^{k+1}$ the coloring

$$\Lambda_G(F_s) = \begin{cases} e_j, & \text{if } s = i_j, \\ e_{k+1}, & \text{otherwise} \end{cases}$$

is induced by a $\mathcal{C}(n, k+1)$-subcomplex $\mathcal{C}(P, \Lambda_G)$ and gives a subgroup $H_G \simeq \mathbb{Z}_2^{m-k-1} \subset \mathbb{Z}_2^m$ such that

$$N(P, \Lambda) \simeq \mathbb{R}\mathcal{Z}_P / H_G \simeq S^n.$$

For $P = \Delta^{n_1} \times \dots \Delta^{n_k}$ this gives D. Gugnin's construction (2019) of the action of $\mathbb{Z}_2^{k-1}$ on $S^{n_1} \times \dots \times S^{n_k}$ with the orbit space $S^{n_1 + \dots + n_k}$.

Hamiltonian subcomplexes

- We call a subcomplex $C = \mathcal{C}(P, c) \subset \partial P$ Hamiltonian, if each q-skeleton of P belongs to the $(q+1)$-skeleton of C.
- By a defining face of a subcomplex $C \subset \partial P$ we call each $(n-2)$-face $F_{i_1} \cap F_{i_2}$, where F_{i_1} and F_{i_2} lie in the same facet of C.

Proposition

A subcomplex $C \subset \partial P$ is Hamiltonian if and only if any two defining $(n-2)$-facets are disjoint.

Vector-coloring induced by a Hamiltonian $\mathcal{C}(n,k)$-subcomplex (Construction H)

- Let $C = \mathcal{C}(P, c) \subset \partial P$ be a proper Hamiltonian $\mathcal{C}(n,k)$-subcomplex.
- For any facet $\widetilde{G}$ of C the adjacency graph of facets F_i of P lying in $\widetilde{G}$ is a tree, which induces a coloring χ of facets of P in two colors 0 and 1.
- Define a linearly independent coloring of rank $k+1$

$$\widetilde{\Lambda}_C(F_i) = \begin{cases} a_{c(i)} & \text{if } \chi(F_i) = 1, \\ b_{c(i)} & \text{if } \chi(F_i) = 0, \end{cases}$$

  where $\{a_1, \dots, a_k, b_1\}$ is a basis in $\mathbb{Z}_2^{k+1}$ and $a_i + b_i = \tau \ \forall i$.
- We call $\widetilde{\Lambda}_C$ a vector-coloring induced by C.
- It is defined up to a change of coordinates in $\mathbb{Z}_2^{k+1}$.

The involution $\tau \in \mathbb{Z}_2^{k+1}$ acting on $N(P, \widetilde{\Lambda}_C)$ is hyperelliptic.

Example for dimension $n = 3$

- Let Γ be a Hamiltonian empty cycle, theta-graph or a K_4-subgraph in ∂P.
- For each connected component of $\partial P \setminus \Gamma$ the incidence graph of the facets of P from this component is a tree and can be colored in two colors 0 and 1.
- If at least one component is not a facet, then we obtain a linearly independent coloring $\widetilde{\Lambda}_\Gamma$ and a hyperelliptic involution τ_Γ on $N(P, \widetilde{\Lambda}_\Gamma)$ induced by Γ.

Example

If P is a right-angled polytope in $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{L}^3$, $\mathbb{L}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$, then we obtain examples constructed by A.D.Mednykh and A.Yu.Vesnin.

Hyperelliptic manifolds and branched 2-sheeted covers

- Let $C \subset \partial P$ be a proper Hamiltonian $\mathcal{C}(n, k)$-subcomplex and $M = M_1^{n-2} \sqcup \dots \sqcup M_s^{n-2}$ be the set of its defining faces.
- There are two colorings Λ_C and $\widetilde{\Lambda}_C$ such that $N(P, \Lambda_C) \simeq S^n$ and $N(P, \widetilde{\Lambda}_C)$ is a hyperelliptic manifold with the projection $\pi\colon N(P, \widetilde{\Lambda}_C) \to N(P, \Lambda_C)$.

- outside the preimages of M the mapping φ is a 2-sheeted covering;
- on the preimages of M it locally has the form $(z, x) \to (z^2, x)$, where M corresponds to the points $(0, x)$

In particular, π is a branched 2-sheeted covering with the branch set the preimage of M.

Structure of the branch set

- For each defining $(n-2)$-face M_i let r_i be the number of facets $\widetilde{G}_j$ of C such that $\widetilde{G}_j \cap M \neq \varnothing$ and $M \not\subset \widetilde{G}_j$.
- Define

$$N(M_i) = M_i \times \mathbb{Z}_2^{r_i} / \sim,$$

where $(p, t) \sim (q, s) \Leftrightarrow p = q, t - s \in \langle e_j : p \in M_i \cap \widetilde{G}_j \rangle$, and $e_1, \ldots, e_{r_i}$ is the standard basis.

The preimage of M_i is a disjoint union of 2^{k+1-r_i} copies of the closed orientable $(n-2)$-manifold $N(M_i)$

.

Classification of hyperelliptic involutions for $n \leqslant 4$

Theorem (E.,24)

For $n \leqslant 4$ hyperelliptic involutions in $\mathbb{Z}_2^r = \mathbb{Z}_2^m/H$ acting on $N(P, \Lambda)$ are in bijection with Hamiltonian $\mathcal{C}(n, r-1)$-subcomplexes inducing Λ.

Corollary (E., 24)

Let $N(P, \Lambda)$ be a small cover over a 3-polytope P. Then there are at most three hyperelliptic involutions in $\mathbb{Z}_2^3$. There are three of them iff any of the following equivalent conditions holds

- $N(P, \Lambda)/\tau \simeq S^3$ for any orientation preserving involution $\tau \in \mathbb{Z}_2^3$ (for this formulation thanks to Vladimir Gorchakov);
- $N(P, \Lambda)$ is a rational homology 3-sphere;
- Λ is induced by three Hamiltonian cycles on P such that any edge of P belongs to exactly two of them.

Corollary: Hyperelliptic small covers

- Each Hamiltonian $\mathcal{C}(n, n-1)$-subcomplex $C \subset \partial P$ induces a linearly independent vector-coloring $\widetilde{\Lambda}_C$ of rank n and a hyperelliptic involution $\tau_C \in \mathbb{Z}_2^n$ on the small cover $N(P, \Lambda_C)$.
- For $n \leqslant 4$ and a small cover $N(P, \Lambda)$ hyperelliptic involutions in $\mathbb{Z}_2^n$ are in bijection with Hamiltonian $\mathcal{C}(n, n-1)$-subcomplexes $C \subset \partial P$ inducing Λ.
- For any Hamiltonian $\mathcal{C}(n, n-1)$-subcomplex $C \subset \partial P$ the vertices of P lie on a disjoint set of $(m-n+1)$ defining $(n-2)$-faces of C. Moreover, each defining face has a coloring in $(n-2)$-colors ($\Leftrightarrow$ all 2-faces are even-gons), and all the edges of P transversal to defining faces form a Hamiltonian cycle.

Hyperelliptic small covers over 4-polytopes

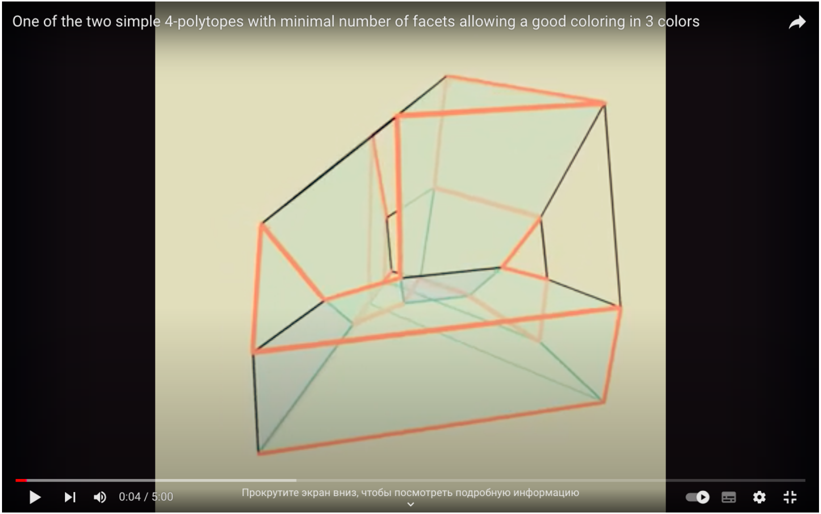
Proposition (E., 24)

A 4-dimensional small cover $N(P,\Lambda)$ has a hyperelliptic involution in $\mathbb{Z}_2^4$ if and only if Λ is defined by a Hamiltonian cycle Γ on P that is the intersection of three unions of facets of P such that

- each union is a 3-ball;
- each facet of P belongs to exactly one union.

- The first example of such structures was recently found by Alexei Koretskii.
- He found a polytope with 9 facets obtained from the 4-simplex by a sequence of four truncations of a vertex and 2-faces. Nine is the minimal possible number of facets.
- He proved also that such structures exist on the polytopes obtained from the 4-cube I^4 and $\Delta^3 \times I$ by cutting off all vertices, then all old edges, then all old 2-faces. $(\partial P^* = (\partial(I^4)^*)'$ or $(\partial(\Delta^3 \times I)^*)')$

The polytope found by Alexei Koretskii



The following result gives an answer to the question posed by A.D. Mednykh on the seminars.

Question (A.D. Mednykh)

For which 4-dimensional geometries there is a right-angled polytope P and a linearly independent vector coloring Λ such that $N(P, \Lambda)$ has a hyperelliptic involution in $\mathbb{Z}_2^m / H(\Lambda)$?

Any such a coloring gives a geometric hyperelliptic manifold generalizing 3-dimensional examples.

Geometric hyperelliptic 4-manifolds

Theorem (E.,24)

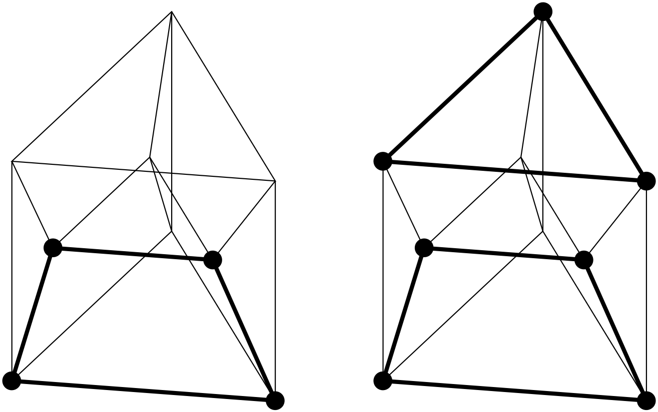
There are 4-dimensional geometric hyperelliptic manifolds with geometries

- $\mathbb{S}^4$ (from Δ^3);
- $\mathbb{S}^3 \times \mathbb{R}$ (from $\Delta^3 \times \mathrm{I}$);
- $\mathbb{S}^2 \times \mathbb{S}^2$ (from $\Delta^2 \times \Delta^2$);
- $\mathbb{S}^2 \times \mathbb{R}^2$ (from $\Delta^2 \times \mathrm{I}^2$);
- $\mathbb{S}^2 \times \mathbb{L}^2$ (from $\Delta^2 \times$ k-gon, $k \geqslant 5$);
- $\mathbb{L}^2 \times \mathbb{L}^2$ (from 5-gon $\times$ 5-gon);

There are no linearly independent colorings of right-angled polytopes in geometries $\mathbb{R}^4$, $\mathbb{R}^2 \times \mathbb{L}^2$, $\mathbb{L}^3 \times \mathbb{R}$ and $\mathbb{L}^4$ such that the manifold $\mathrm{N}(\mathrm{P}, \Lambda)$ has a hyperelliptic involution in $\mathbb{Z}_2^m / \mathrm{H}(\Lambda)$.

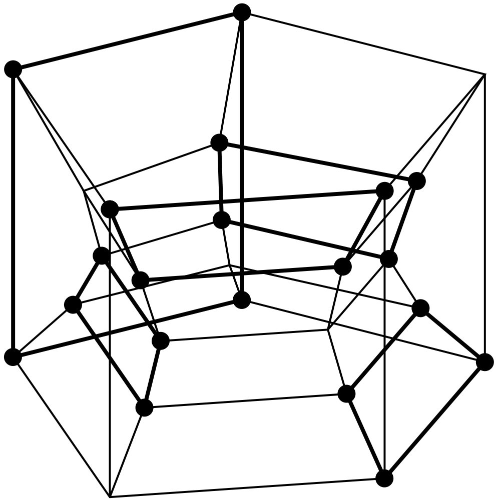
This exhausts all 4-geometries that are products of Eucledean, spherical and hyperbolic geometries.

The case of $\mathbb{S}^2 \times \mathbb{S}^2$



Defining faces of unique (up to symmetries) Hamiltonian $\mathcal{C}(4,5)$- and $\mathcal{C}(4,4)$-subcomplexes in $\Delta^2 \times \Delta^2$.
The branch sets are T^2 and $\mathrm{S}^2 \sqcup \mathrm{T}^2$.

The case of $\mathbb{L}^2 \times \mathbb{L}^2$



Defining faces of a unique (up to symmetries) Hamiltonian $\mathcal{C}(4,5)$-subcomplex in 5-gon × 5-gon. The branch set is $\bigsqcup_5 \mathrm{T}^2$.

Negative results: $\mathbb{R}^n$, $\mathbb{L}^n$, $n \geqslant 4$, $\mathbb{L}^3 \times \mathbb{R}$, $\mathbb{L}^2 \times \mathbb{R}^2$

- The n-cube I^n does not admit Hamiltonian $\mathcal{C}(n, k)$-subcomplexes for $n > 3$. In particular, the geometry $\mathbb{R}^n$, $n > 4$ does not arise in Construction H.
- If a 4-polytope P admits a Hamiltonian $\mathcal{C}(4, r)$-subcomplex, then P has at least one triangular or quadrangular 2-face. In particular, $\mathbb{L}^4$ does not not arise in Construction H.
- For $n > 4$ due to V.V. Nikulin's results (1982) there are no compact right-angled polytopes in $\mathbb{L}^n$, so this geometry also does arise in Construction H.
- If a 4-polytope $P = Q \times I$ admits a Hamiltonian $\mathcal{C}(4, r)$-subcomplex, then at least one of the defining 2-faces of P is a triangle. Thus, $\mathbb{L}^3 \times \mathbb{R}$, $\mathbb{L}^2 \times \mathbb{R}^2$, $\mathbb{R}^4$ do not arise in Construction H.

Positive results: $\mathbb{S}^n$

- For $n \geqslant 4$ the simplex Δ^n up to symmetries has a unique proper Hamiltonian $\mathcal{C}(n, r)$-subcomplex defined by a single $(n-2)$-face Δ^{n-2}. For this subcomplex $r = n$.
- It corresponds to a hyperelliptic involution on the manifold $\mathbb{R}\mathcal{Z}_{\Delta^n} \simeq S^n$. In particular, the geometry $\mathbb{S}^n$ arises in Construction H.
- The branch set of the covering $S^n \to S^n$ is the sphere S^{n-2}.

Positive results: $\mathbb{S}^p \times \mathbb{S}^q$

- For $n \geqslant 4$ the prism $\Delta^{n-1} \times I$ up to symmetries admits exactly 3 Hamiltonian $\mathcal{C}(n, r)$-subcomplexes: a unique $\mathcal{C}(n, n+1)$-subcomplex and two $\mathcal{C}(n, n)$-subcomplexes. In particular, the geometry $\mathbb{S}^{n-1} \times \mathbb{R}$ arises in Construction H. For the $\mathcal{C}(n, n+1)$-subcomplex and the $\mathcal{C}(n, n)$-subcomplexes the branch sets of the coverings $S^{n-1} \times S^1 \to S^n$ and $N(P, \widetilde{\Lambda}_C) \to S^n$ are disjoint unions of two spheres S^{n-2}.
- For any $p, q \geqslant 1$ the face $\Delta^{p-1} \times \Delta^{q-1}$ defines a Hamiltonian $\mathcal{C}(n, n+1)$-subcomplex in $P^n = \Delta^p \times \Delta^q$. In particular, for $p, q \geqslant 2$ the geometry $\mathbb{S}^p \times \mathbb{S}^q$ arises in Construction H. For $p, q \geqslant 2$ the branch set of the coverings $S^p \times S^q \to S^n$ is homeomorphic to $S^{p-1} \times S^{q-1}$.

Positive results: $\mathbb{S}^p \times \mathbb{S}^q \times \mathbb{R}$, $\mathbb{S}^p \times \mathbb{R}^2$ and $\mathbb{S}^p \times \mathbb{L}^2$

- For any $p, q \geqslant 1$ the polytope $P^n = \Delta^p \times \Delta^q \times I$ has a Hamiltonian $C(n, n+1)$-subcomplex defined by two faces $\Delta^{p-1} \times \Delta^q \times \{0\}$ and $\Delta^p \times \Delta^{q-1} \times \{1\}$. In particular, the geometries $\mathbb{S}^p \times \mathbb{S}^q \times \mathbb{R}$, $p, q \geqslant 2$, $\mathbb{S}^p \times \mathbb{R}^2$, $p \geqslant 2$, and $\mathbb{R}^3$ arise in Construction H. The branch set of the covering is $S^{p-1} \times S^q \sqcup S^p \times S^{q-1}$.
- Let P_k be a k-gon, $k \geqslant 3$. Then for $p \geqslant 1$ the polytope $P^n = \Delta^p \times P_k$ admits Hamiltonian $\mathcal{C}(n, n)$- and $\mathcal{C}(n, n+1)$-subcomplexes C_n and C_{n+1}. In particular, taking a right-angled triangle in $\mathbb{S}^2$, a square in $\mathbb{R}^2$ and a right-angled k-gon, $k \geqslant 5$, in $\mathbb{L}^2$ we see that the geometries $\mathbb{S}^p \times \mathbb{S}^2$, $\mathbb{S}^p \times \mathbb{R}^2$ and $\mathbb{S}^p \times \mathbb{L}^2$, $p \geqslant 2$, arise in Construction H. The branch set of the covering is $S^{n-3} \times S^1 \bigsqcup_{2(k-3)} S^{n-2}$ for C_{n+1}, and $S^{n-3} \times S^1 \bigsqcup_{k-2} S^{n-2}$ for C_n.

Negative result: $\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_k}$, $k \geqslant 4$ or $k \geqslant 3$ and $n_i \geqslant 2$

Proposition

The polytope $P = \Delta^{n_1} \times \cdots \times \Delta^{n_k}$ admits a proper Hamiltonian $\mathcal{C}(n, k)$-subcomplex if and only if P is one of the following polytopes:

1. Δ^n, $n \geqslant 2$.
   Possible subcomplexes are $\mathcal{C}(2,1)$, $\mathcal{C}(2,2)$, $\mathcal{C}(3,2)$, $\mathcal{C}(3,3)$, $\mathcal{C}(n,n)$, $n \geqslant 4$.
2. $\Delta^p \times \Delta^q$, $1 \leqslant p \leqslant q$.
   Possible subcomplexes are $\mathcal{C}(2,1)$, $\mathcal{C}(2,2)$, $\mathcal{C}(2,3)$, $\mathcal{C}(3,2)$, $\mathcal{C}(3,3)$, $\mathcal{C}(3,4)$, $\mathcal{C}(n,n)$ (for $p = 2$) and $\mathcal{C}(n, n+1)$, $n \geqslant 4$.
3. $\Delta^p \times \Delta^q \times I$, $1 \leqslant p \leqslant q$.
   Possible subcomplexes are $\mathcal{C}(3,2)$, $\mathcal{C}(3,3)$, $\mathcal{C}(3,4)$, $\mathcal{C}(n,n)$ (for $p = 1$) and $\mathcal{C}(n, n+1)$, $n \geqslant 4$.

Classification of $\mathcal{C}(n, k)$-subcomplexes in (k-gon)×(l-gon)

Proposition

The 4-polytope (k-gon)×(l-gon) admits a Hamiltonian $\mathcal{C}(4, k)$-subcomplex if and only if P is one of the following polytopes:

1. $\Delta^2 \times$(l-gon), $l \geqslant 3$.
   Up to symmetries all $\mathcal{C}(4, k)$-subcomplexes are reduced to a unique $\mathcal{C}(4, 4)$-subcomplex and a unique $\mathcal{C}(4, 5)$-subcomplex in $\Delta^2 \times \Delta^2$.
2. (5-gon)×(5-gon).
   Up to symmetries there is a unique $\mathcal{C}(4, k)$-subcomplex, namely the $\mathcal{C}(4, 5)$-subcomplex described above.

Negative results: products of $\geqslant 3$ geometries

Proposition

Let $P^n = P_1 \times \cdots \times P_k$, where $k \geqslant 3$ and P_i are flag polytopes or simplices of dimensions $n_i \geqslant 2$. Then P does not have Hamiltonian $\mathcal{C}(n, r)$-subcomplexes.

Corollary

Geometries $\mathbb{X} = \mathbb{X}_1^{n_1} \times \cdots \times \mathbb{X}_k^{n_k}$, where $n_i \geqslant 2$, and $\mathbb{X}_i^{n_i}$ is $\mathbb{S}^{n_i}$, $\mathbb{R}^{n_i}$, $\mathbb{L}^2$, or $\mathbb{L}^3$. Then $\mathbb{X}$ does not admit Construction H.

Conjecture

If $\mathbb{X} = \mathbb{X}_1^{n_1} \times \cdots \times \mathbb{X}_k^{n_k}$ admits Construction H, then $k = 3$ and $\mathbb{X}_i^{n_i} = \mathbb{R}$ for some i.

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