Markov theorem for doodles on two-sphere

Konstantin Gotin

Sobolev Institute of Mathematics

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Link is an embedding of collection of circles in R considered up to isotopy of $\mathbb{R}^3$.

Link can be considered as set of planar diagrams related by Redemeister moves.

Theorem (J. Alexander)

Any link is closure of some braid.

Theorem (A. Markov)

There are three necessary and sufficient conditions for two braids to have equivalent closures:

1) they are equivalent braids,
2) they are conjugate braids,
3) appending or removing on the right of the braid a strand which crosses strand to its left exactly once.

Doodle diagram is an immersion of collection disjoint circles to S^2 with no triple or higher intersection points. We assume that number of double points of doodle diagram is finite.

Doodle is class of doodle diagrams related by isotopies of S^2 and following moves:

A twin diagram is a configuration of n arcs in $\mathbb{R} \times [0,1]$ such that:

1) for any $\mathrm{i} = 1, \ldots, \mathrm{n}$ there is unique $\mathrm{j} = 1, \ldots, \mathrm{n}$ such that $(\mathrm{i}, 0)$ and $(\mathrm{j}, 0)$ are connected by a curve,
2) any curve is monotonic by y-coordinate,
3) the number of double points is finite and there no triple or higher intersection points.

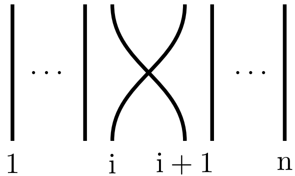
Two twin diagrams are equivalent if they can be related by a finite sequence of moves R_2 and isotopies of $\mathbb{R} \times (0,1)$ such that conditions 1), 2), 3) are satisfied.

The product of two twins τ_1 and τ_2 on the same number of strands is defined by putting diagram of τ_1 on top of the diagram of τ_2 and squeezing along y-coordinate.

Theorem (M. Khovanov)

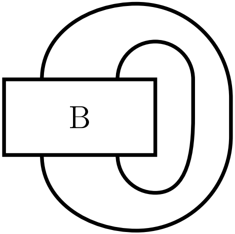
A set of twins form a group, denoted by $\mathrm{TW_n}$, generated by elements $\mathrm{s_1, s_2, \ldots, s_{n-1}}$ presented in figure below which satisfy the following relations:

$$\mathrm{s_i^2 = e}, \quad \text{for } \mathrm{i = 1, \ldots, n-1},$$
$$\mathrm{s_i s_j = s_j s_i}, \quad \text{if } \mathrm{|i-j| > 1}$$



Theorem (M. Khovanov)

Any doodle can be presented by following diagram,



here B is diagram of some twin.

Theorem

If two twins has equivalent closures they are related by sequence of following moves and its inverses:

M_1: $\beta \otimes I \leftrightarrow I \otimes \beta$;

M_2: $\beta \to \alpha\beta\alpha^{-1}$, for $\alpha \in \mathrm{TW_n}$;

M_3: $\beta \to (I \otimes \beta)s_1 s_2 \ldots s_{i-1} s_i s_{i-1} \ldots s_2 s_1$;

M_4: $\beta \to (\beta \otimes I)s_n s_{n-1} \ldots s_{i+1} s_i s_{i+1} \ldots s_{n-1} s_n$;

here $\beta \in \mathrm{TW_n}$, $s_i \in \mathrm{TW_{n+1}}, i = 1, \ldots, n$.

Thank you for attention!