Non-abelian tensor product and orderability of groups

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Theorem (Schur, 1904): Consider a central extension

$$1 \longrightarrow K \longrightarrow H \longrightarrow G \longrightarrow 1$$

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Ways to generalize Baer theorem:

- Consider n-central extensions and relate properties of G to properties of $\gamma_{n+1}(H)$.
- Change lower central series to some other normal series.

Tensor product of groups

Let G and H be groups acting on each other. We require that the actions are compatible:

$$^{(^{g}h)}g' = {}^{ghg^{-1}}g' \qquad ^{(^{h}g)}h' = {}^{hgh^{-1}}h'$$

$$\text{where } ghg^{-1}, hgh^{-1} \in G * H.$$

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where $ghg^{-1}, hgh^{-1} \in G * H$.

Tensor product of G and H is a group $G \otimes H$, freely generated by symbols $g \otimes h$ with relations

$$gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h)$$

$$g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h')$$

In the case $G = H$ and the action is conjugation in G tensor product is called tensor square.

An exterior square $G \wedge G$ is defined as $\dfrac{G \otimes G}{\langle g \otimes g \rangle}$.

Crossed pairing

A map φ from the set $G \times H$ to a group L is called a crossed pairing if

$$\varphi(gg', h) = \varphi({}^{g}g', {}^{g}h)\varphi(g, h)$$

$$\varphi(g, hh') = \varphi(g, h)\varphi({}^{h}g, {}^{h}h')$$

A crossed pairing φ determines a homomorphism of groups

$$\varphi^* : \ G \otimes H \to L, \quad \varphi^*(g \otimes h) = \varphi(g, h).$$

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Example:

Let $H = G$ and consider a commutator map

$$\varphi(g_1, g_2) = [g_1, g_2].$$

Commutator identities imply that this is a crossed pairing and we have a homomorphism $\varphi^* : \; G \otimes G \to G$ that sends $g_1 \otimes g_2$ to $[g_1, g_2]$.

Examples of crossed pairings

Since $\varphi^*(g \otimes g) = 1$ it also defines a homomorphism $G \wedge G \to G$.

Theorem (Brown, Loday, 1987): $\ker(G \wedge G \to G) \cong H_2(G)$.

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Theorem (Brown, Loday, 1987): $\ker(G \wedge G \to G) \cong H_2(G)$.

$$G = F/R, \quad G \wedge G \cong \frac{[F,F]}{[F,R]}$$

$$1 \longrightarrow \frac{R \cap [F,F]}{[F,R]} \longrightarrow \frac{[F,F]}{[F,R]} \longrightarrow \frac{[F,F]}{R \cap [F,F]} \longrightarrow 1$$

$$\frac{R \cap [F,F]}{[F,R]} \cong H_2(G) \text{ (Hopf's formula)}$$

Consider an arbitrary central extension

$$1 \longrightarrow A \longrightarrow K \xrightarrow{\pi} G \longrightarrow 1$$

The commutator maps lifts to the map

$$\Phi(g, h) = [\bar{g}, \bar{h}] \text{ where } \pi(\bar{g}) = g,\ \pi(\bar{h}) = h$$

which is a crossed pairing. This defines homomorphisms

$$g \otimes h \mapsto [\bar{g}, \bar{h}] \text{ and } g \wedge h \mapsto [\bar{g}, \bar{h}]$$

To prove Schur's theorem it is sufficient to show that $G \wedge G$ is finite whenever G is finite.

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$$1 \longrightarrow H_2(G) \longrightarrow G \wedge G \longrightarrow \gamma_2(G) \longrightarrow 1$$

- $\gamma_2(G)$ is a subgroup of finite group
- Homology groups of finite groups are finite.
- Extensions of finite groups by finite are finite.

When G and H are endowed with trivial actions we have

$$G \otimes H \cong G_{ab} \otimes_{\mathbb{Z}} H_{ab}.$$

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Remark: Tensor product is not abelian in general. Let $G = H = F$ – free group on n elements.

$$1 \longrightarrow \text{Ker } \pi \longrightarrow F \otimes F \xrightarrow{\pi} [F, F] \longrightarrow 1$$

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It is often the case that properties of a tensor product and the groups are similar.

- if G and H are finite, then $G \otimes H$ is finite.
- if G and H are perfect groups, then $G \otimes H$ is perfect

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
 & \Gamma(G_{ab}) \arrow[r] \arrow[d] & J(G) \arrow[r] \arrow[d] & H_2(G) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \nabla(G) \arrow[r] \arrow[d] & G \otimes G \arrow[r] \arrow[d] & G \wedge G \arrow[r] \arrow[d] & 1 \\
 & 1 \arrow[r] & G' \arrow[r] \arrow[d] & G' \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 &
\end{tikzcd}$$

$J(G) := \pi_3(\Sigma K(G,1))$,
$\Gamma(G_{ab})$ is a quadratic Whitehead functor.
$\nabla(G) = \langle g \otimes g \rangle \subset G \otimes G$.

There are well defined actions of G on $G \otimes G$

$$^{g_3}(g_1 \otimes g_2) = {}^{g_3}g_1 \otimes {}^{g_3}g_2,$$

and $G \otimes G$ on G given by

$$^{g_1 \otimes g_2}g_3 = {}^{[g_1, g_2]}g_3.$$

These actions are compatible and we can define

$$G^{\otimes 3} := (G \otimes G) \otimes G.$$

For $n \geqslant 3$ we can inductively define the n-fold tensor product $G^{\otimes n}$, by considering the actions of G and $G^{\otimes n-1}$ on each other defined by

$$\begin{aligned} &{}^{g_n}(\dots((g_1 \otimes g_2) \otimes \dots \otimes g_{n-2}) \otimes g_{n-1}) = \\ &= (\dots(({}^{g_n}g_1 \otimes {}^{g_n}g_2) \otimes \dots \otimes {}^{g_n}g_{n-2}) \otimes {}^{g_n}g_{n-1}) \end{aligned}$$

and

$${}^{(\dots((g_1 \otimes g_2) \otimes \dots \otimes g_{n-2}) \otimes g_{n-1})}g_n = {}^{[\dots[[g_1, g_2], g_3], \dots, g_{n-1}]}g_1.$$

And there is a well-defined homomorphism $\lambda_n^G : G^{\otimes n} \to \gamma_n(G)$ defined on generators by

$$(\dots(g_1 \otimes g_2) \otimes \dots \otimes g_{n-1}) \otimes g_n \mapsto [\dots[g_1, g_2], \dots, g_n].$$

For any *n*-central extension

$$1 \longrightarrow N \longrightarrow H \longrightarrow G \longrightarrow 1$$

There is an epimorphism $G^{\otimes n+1} \to \gamma_n(H)$ that makes the diagram commute.

$$\begin{tikzcd}
G^{\otimes n+1} \arrow[r] \arrow[d] & G^{\otimes n+1} \arrow[d, "\lambda^{G}_{n+\mathbf{1}}"] \\
\gamma_{n+1}(H) \arrow[r] & \gamma_{n+1}(G)
\end{tikzcd}$$

To prove Baer's theorem it is sufficient to prove that $G^{\otimes n}$ is finite. For derrived series one can consider *n*-fold wedge product

$$G^{\wedge(n)} = G^{\wedge(n-1)} \wedge G^{\wedge(n-1)}, \quad n > 1$$

A group G is left-orderable if there is a left-invariant linear order on G

$$g_1 < g_2 \Leftrightarrow hg_1 < hg_2.$$

- Subgroup of a left-orderable group is left-orderable.
- Left-ordered groups are closed with respect to taking extensions.
- Left-ordered groups are torsion-free.

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Examples of orderable groups: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, F_n, B_n, \pi_1(\Sigma_G)$, $\pi_1(M)$ where M — compact, connected, $\mathbb{P}^2$- irreducible 3-manifold with nontrivial homomorphism to an orderable group.

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Non-examples of orderable groups: $\mathbb{Z}_n$, $\mathbb{Q}/\mathbb{Z}$, $\mathrm{PSL}(2, \mathbb{R})$, any group with torsion, Hantzsche–Wendt group $\langle x, y \mid x^{-1}y^2xy^2, y^{-1}x^2yx^2 \rangle$.

Group G is circularly orderable if there is a function $c : G^3 \to \{0, \pm 1\}$ s.t.

1. $c^{-1}(0) = \{(g_1, g_2, g_3) \in G^3 \mid g_i = g_j \text{ for some } i \neq j\}$
2. $c(g_2, g_3, g_4) - c(g_1, g_3, g_4) + c(g_1, g_2, g_4) - c(g_1, g_2, g_3) = 0$
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Non-examples of circularly orderable groups: finite non cyclic groups.

A circular order $c: \; G^3 \to \{0, \pm 1\} \subset \mathbb{Z}$ is a homogenous 2-cocycle.
Theorem (S. Zeleva, 1975): There is a 1-1 correspondence between homogenous 2-cocycles as above and inhomogenous 2-cocycles satisfying

1. $f(g,h) \in \{0,1\}$
2. $f(g,g^{-1}) = 1$ for all $g \neq \mathrm{id}$

$$f^{(c)}(g,h) = \begin{cases} 0 & \text{if } g = id \text{ or } h = id, \\ 1 & \text{if } gh = id \text{ and } g \neq id \\ \frac{1}{2}(1 - c(id, g, gh)) & \text{otherwise.} \end{cases}$$

$$c^{(f)}(g_1, g_2, g_3) = \begin{cases} 0 & \text{if } g_i = g_j \text{ for some } i \neq j \\ 1 - 2f\left(g_1^{-1} g_2, g_2^{-1} g_3\right) & \text{otherwise} \end{cases}$$

Consider a central extension $\widetilde{G}_f$, associated to $[f]$, which is constructed by equipping the set $\mathbb{Z} \times G$ with the operation

$$(a, g)(b, h) = (a + b + f(g, h), gh)\,.$$

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An element $g \in G$ is called $<$-cofinal whenever the cyclic subgroup $\langle g \rangle$ is $<$-cofinal as a set.

The central element $(1, id) \in \widetilde{G_f}$ is positive and $<_c$-cofinal in the left-ordering $<_c$ of $\widetilde{G_f}$ and the construction is reversible.

Suppose that G is a left-ordered group with ordering $<$, and there is a central element $z \in G$ which is positive and $<$-cofinal. Then the quotient $G/\langle z\rangle$ inherits a circular ordering.

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- If $[f] = 0 \in H^2(G)$, then circular orderability implies orderability.
- If $[f] \in H^2(G; \mathbb{Z})$ has order k, then G contains a left-orderable normal subgroup H such that $G/H \cong \mathbb{Z}/k\mathbb{Z}$.

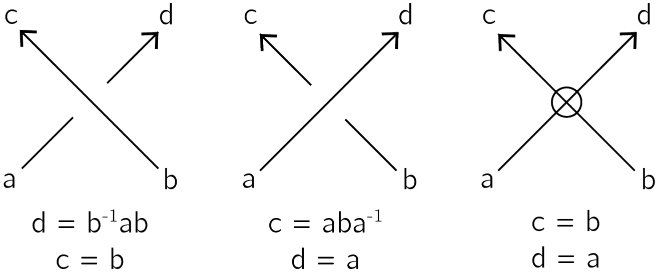
$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
 & \Gamma(G_{ab}) \arrow[r] \arrow[d] & J(G) \arrow[r] \arrow[d] & H_2(G) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \nabla(G) \arrow[r] \arrow[d] & G \otimes G \arrow[r] \arrow[d] & G \wedge G \arrow[r] \arrow[d] & 1 \\
 & 1 \arrow[r] & G' \arrow[r] \arrow[d] & G' \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 &
\end{tikzcd}$$

Let G be a circularly orderable group

- If $H_2(G)$ is torsion-free, then $G \wedge G$ is orderable.
- If $H_2(G)$ is torsion-free, $H_1(G)$ is finitely generated torsion free, then $G \otimes G$ is orderable.

Virtual Knot Groups

By G_K we understand a group, with generators corresponding to arcs and realtions defined by crossings



$d = b^{-1}ab$
$c = b$

$c = aba^{-1}$
$d = a$

$c = b$
$d = a$

Рис. 1: Relations for each crossing

Deficiency of a group presentaion is equal to # generators $-$ # relators.

Theorem (Se-Goo Kim, 1999): Virtual knot groups are precisely groups defined by a Wirtinger presentation of deficiency 0.

It follows that $H_2(G_K)$ is a cyclic group and for virtual knot groups with finite $H_2(G_K)$ circular orderability coincides with orderability.

A. Ichimori computed groups of virtual knots in Jeremy Green's table and showed, that each group is isomorphic either to an infinite cyclic group or to some group G_i from the following list.

$$
\begin{aligned}
G_1 &= \langle x, y \mid xyx = yxy \rangle \\
G_2 &= \left\langle x, a \mid x^{-1}ax = a^{-1}, a^3 = 1 \right\rangle, \\
G_3 &= \left\langle x, y \mid xyx = yxy, y = x^{-3}yx^3 \right\rangle, \\
G_4 &= \left\langle x, y \mid x^{-1}yxy^{-1}x = yx^{-1}yxy^{-1} \right\rangle, \\
G_5 &= \left\langle x, a \mid x^{-1}ax = a^{-1}, a^5 = 1 \right\rangle \\
G_6 &= \left\langle x, a \mid x^{-1}ax = a^2, a^5 = 1 \right\rangle, \\
G_7 &= \left\langle x, y \mid x^{-1}yx = y^{-1}xy \right\rangle \\
G_8 &= \left\langle x, a \mid x^{-1}ax = a^2, a^7 = 1 \right\rangle, \\
G_9 &= \left\langle x, y \mid xyxy^{-1}x^{-1} = yx^{-1}yxy^{-1}, x^{-1}y^{-1}xyx = y^{-1}x^{-1}yxy \right\rangle.
\end{aligned}
$$

Theorem: Groups G_1, G_4, G_7, G_9 are orderable. Groups G_2, G_3, G_5, G_6, G_8 are not circulary orderable.

Theorem(Donadze, García-Martínez, 2021):

1. There is an epimorphism of groups

$$\mathrm{Ker}\left\{\lambda_{k+1}^{G} : G^{\otimes k+1} \to G\right\} \to \frac{R \cap \gamma_{k+1}(F)}{\gamma_{k+1}(R, F)}$$

2. There is an isomorphism of groups

$$\mathrm{Ker}\left\{\mu_{k+1}^{G} : G^{\wedge(k+1)} \to G\right\} \to \frac{R \cap \Gamma_{k+1}(F)}{\Gamma_{k+1}(R, F)}$$

Let G be a group such that $\gamma_n(G) = \gamma_{n+1}(G)$ for some n (such as a virtual knot group). Then

$$\ker(\gamma_n \otimes G \to \gamma_{n+1}(G)) \cong \ker(\gamma_n \wedge G \to \gamma_{n+1}(G)).$$

If $H_3(G/\gamma_n) = 0$ then

$$\ker(\gamma_n \otimes G \to \gamma_{n+1}(G)) \subset H_2(G).$$

In particular if $G/\gamma_n = \mathbb{Z}$ then

$$\ker(\gamma_n \otimes G \to \gamma_{n+1}(G)) \cong H_2(G).$$

An *f*-extension of G is a pair (M, ψ) where M is an extension of G and ψ is a map $\Gamma \to M$ that factorizes f as in diagram below.

$$\begin{tikzcd}
 & & & \Gamma \arrow[dl, "\psi"'] \arrow[dr, "f"] & & \\
1 \arrow[r] & K \arrow[r] & M \arrow[rr] & & G \arrow[r] & 1
\end{tikzcd}$$

An f-extension of G is a pair (M, ψ) where M is an extension of G and ψ is a map $\Gamma \to M$ that factorizes f as in diagram below.

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1 \arrow[r] & K \arrow[r] & M \arrow[rr] & & G \arrow[r] & 1
\end{tikzcd}$$

Assume that f induces a surjection $f_{ab} : \Gamma_{ab} \to G_{ab}$. Define the universal central f-extension of G as a pair $(U, \eta) \in \mathrm{Ext}^{\Gamma}(G; H_2(G, \Gamma))$ corresponding to the identity map in $\mathrm{Hom}(H_2(G, \Gamma), H_2(G, \Gamma))$.

Theorem (Farjoun, Segev, 2016): Assume that the map $f_{ab} : \Gamma_{ab} \to G_{ab}$ induced on the abelianizations is surjective. Then there exists a universal central f-extension (U, η) of G with kernel $H_2(G, \Gamma)$, such that for any central f-extension (E, ψ) of G, there is a unique map of central f-extensions from U to M.

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Choosing apropriate $f : \Gamma \to G$ we have that If $H_2(G, \Gamma)$ is torsion-free, G is circularly orderable, then M is orderable.

Thank you!

Example

$$G_3 = \left\langle x, y \mid xyx = yxy, y = x^{-3}yx^3 \right\rangle$$

Consider a quotient by a central subgroup $\left\langle x^3 \right\rangle$

$$H = G_3/\left\langle x^3 \right\rangle = \left\langle x, y \mid x = y^{-1}x^{-1}yxy, x^3, y^3 \right\rangle$$

Group H is isomorphic to a group $\hat{A}_4 = \left\langle a, b \mid a^3 = b^3 = (ab)^2 \right\rangle$ and isomorphism is given by a map

$$x \mapsto a^2, y \mapsto b^4$$

Since G has a cyclic subgroup of finite index and G is not cyclic, there is a torsion in G. If $H_2(G)$ is finite, then G is not circularly orderable.

Example

Suppose $H_2(G) \cong \mathbb{Z}$, so $G \otimes G$ is left-orderable if G is circularly orderable. We have a natural epimorphism

$$G \otimes G \mapsto H \otimes H$$

$H \otimes H$ is isomorphic to $\mathbb{Z}_3 \times Q$, where Q is the quaternion group. For $G \otimes G$ to be torsion-free it should be a Bieberbach group of dimension ≤ 2 and it is not.