Heegaard splittings of branched cyclic coverings of connected sums of lens spaces

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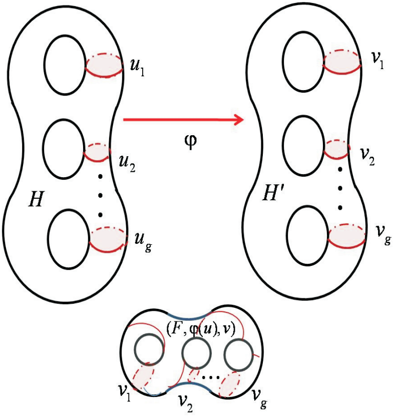
Three - dimensional manifolds

– closed orientable 3-manifolds

– we developed a method for the constructing of 3-manifolds which are branched cyclic coverings of connected sums of lens spaces.

Heegaard diagram

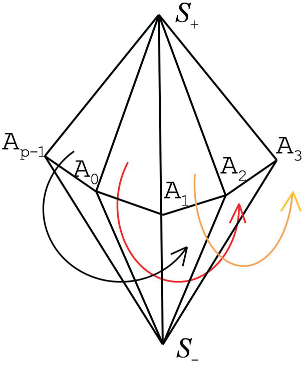
Let $M = H \cup H^{'}$ is a genus g Heegaard splitting of a manifold M, $u = u_1, \ldots, u_g$ and $v = v_1, \ldots, v_g$ are meridian systems for H and $H^{'}$ and $F = \partial H = \partial H^{'}$ is a Heegaard surface. Let $\varphi : F \to F$ be homeomorphism of their boundaries. Then the triple $(F, \varphi(u), v)$ is called a Heegaard diagram of M.

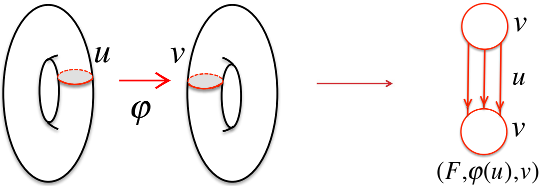


Lens space

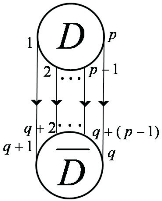
Let $p \geqslant 3$, $0 < q < p$ and $(p, q) = 1$.

Consider a p-gonal bipyramid, i.e. the union of two cones over a regular p-gon, where the vertices of the p-gon are denoted by $A_0, A_1, \ldots, A_{p-1}$ and apex of cones are denoted by S_+ and S_-. For each i we glue the face $A_i S_+ A_{i+1}$ with the face $A_{i+q} S_- A_{i+q+1}$. The manifold obtained is the lens space $L_{p,q}$



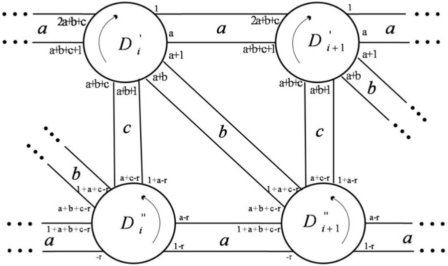


A Heegaard diagram for lens space $L(p, q)$



Branched cyclic coverings

M.J. Dunwoody(1995): introduced some infinite family of diagrams $D(a, b, c, n, r, s)$ with cyclic symmetry, depending on six integer parameters a, b, c, n, r, s, such that $n > 0$ and $a, b, c, r, s \geqslant 0$. Each manifold arising in this way is called a *Dunwoody manifold*.



L. Grasselli , M. Mulazzani (2001): Dunwoody manifolds are exactly the cyclic branched coverings of $(1, 1)$-knots.

Branched cyclic coverings

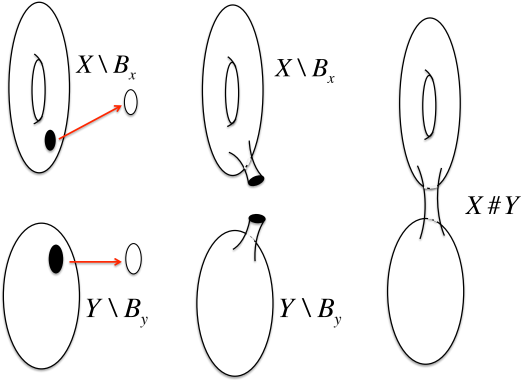
P. Cristofori, M. Mulazzani, A. Vesnin(2007): The existence and uniqueness of the cyclic branched coverings of (g, 1)-knots.

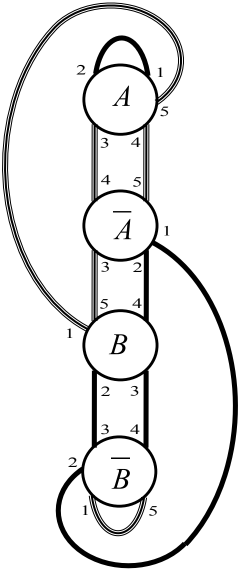
A. Vesnin, T. Kozlovskaya (2011): - considered 3-manifolds from the class of cyclic branched coverings of $(1, b)$-links ($b \geqslant 2$)

P. Cristofori, A. Vesnin, T. Kozlovskaya (2012): - constructed some infinite families of 3-manifolds which are cyclic coverings of lens spaces $L(p, q)$, branched over two-component links

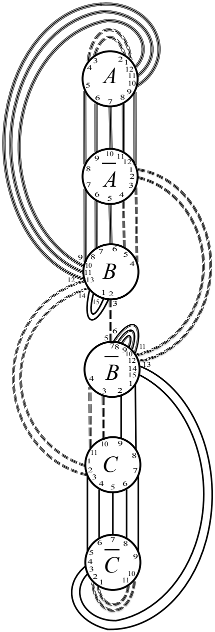
Connected sum of manifolds

Choose the neighborhoods B_x and B_y of points $x \in X$ and $y \in Y$ homeomorphic to an open n - ball. The borders $B_x \setminus X$ and $B_y \setminus Y$ are homeomorphic to an $n-1$ - sphere. Let f be a homeomorphism of boundaries. Then connected sum $X \# Y$ is defined as gluing them together along a homeomorphism f.





A Heegaard diagram for connected sums of two lens spaces $L(3,1)\#L(3,1)$.



A Heegaard diagram for connected sums of three lens spaces $L(8,3)\#L(5,2)\#L(7,2)$

Heegaard diagram

Let $M = H \cup H^{'}$ be a genus g Heegaard splitting of a manifold M and $F = \partial H = \partial H^{'}$ be a Heegaard surface.

Proposition. In order that the curves $u_1, u_2, \ldots, u_g$, v_1,v_2,...,v_g on F form the Heegaard diagram of a certain manifold M it is necessary and sufficient that there hold the following conditions:

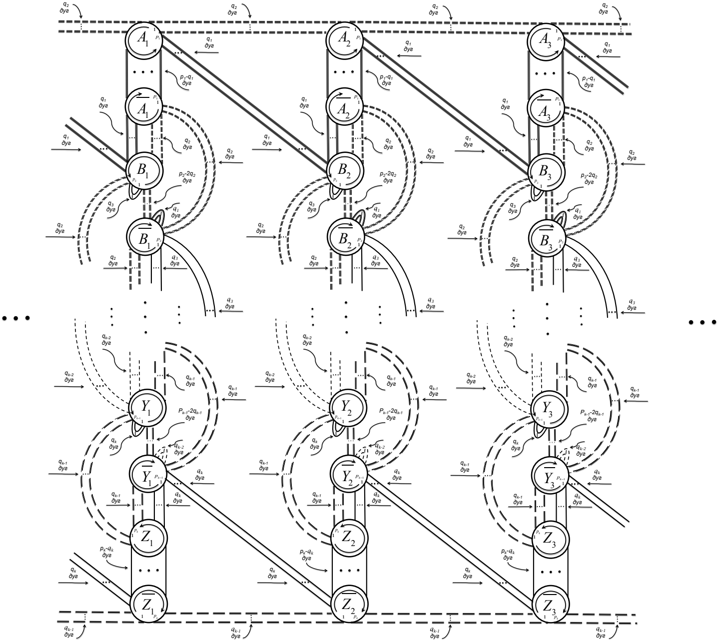
1) the curves u_i do not intersect, and after one makes a cut along them one obtains a connected surface,

2) the same for the curves v_i.

Branched cyclic coverings of connected sums of lens spaces

Theorem.

The diagram presented below is a Heegaard diagram of a closed orientable 3 - manifold which is branched cyclic coverings of connected sums of lens spaces $L(p_1, q_1)\#L(p_2, q_2)\dots\#L(p_k, q_k)$.


A_1
$\overline{A_1}$
B_1
$\overline{B_1}$
Y_1
$\overline{Y_1}$
Z_1
$\overline{Z_1}$
A_2
$\overline{A_2}$
B_2
$\overline{B_2}$
Y_2
$\overline{Y_2}$
Z_2
$\overline{Z_2}$
A_3
$\overline{A_3}$
B_3
$\overline{B_3}$
Y_3
$\overline{Y_3}$
Z_3
$\overline{Z_3}$
q_1 дуг
q_2 дуг
q_3 дуг
p_1-q_1 дуг
p_2-2q_2 дуг
q_{k-2} дуг
q_{k-1} дуг
P_{k-1}-2q_{k-1} дуг
q_k дуг
p_k-q_k дуг