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Groups and quandles
in low-dimensional topology

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MULTI-VIRTUAL BRAID GROUPS

Kozlovskaya Tatyana

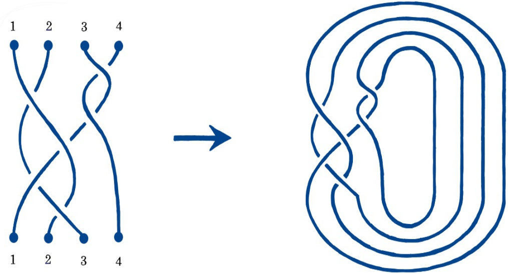
INTRODUCTION

- Valeriy G. Bardakov, Tatyana A. Kozlovskaya, Komal Negi, and Madeti Prabhakar, *Multi-virtual braid groups.*
- Valeriy G. Bardakov, Tatyana A. Kozlovskaya, *Representations of multi-virtual braid group.*

We study multi-virtual braid groups which were introduced by Prof. Kauffman in 2024. We define some of their subgroups and construct representations of multi-virtual braid groups by automorphisms of some groups. We give an answer on a question of Prof. Kauffman on non-triviality of 2-multi-virtual knots.

KNOTS AND BRAIDS

Classical Knot theory (KT) $\longleftrightarrow$ Braid Groups B_n, $n = 1, 2, \ldots$.
By Alexander's theorem every link is a closure of some braid.



4-strand braid β and its closure $\widehat{\beta}$

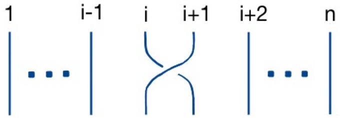
KNOTS AND BRAIDS

Virtual Knot theory (VKT) $\longleftrightarrow$ Virtual Braid Groups VB_n, $n = 1, 2, \ldots$ L. Kauffman (1996).

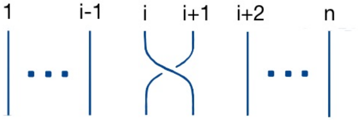
L. Kauffman, *Multi-Virtual Knot Theory*, https://arxiv.org/abs/2409.07499, 82 pp.

Multiple Virtual Knot theory (MVKT) $\longleftrightarrow$ Multiple Virtual Braid Groups MVB_n, $n = 1, 2, \ldots$ L. Kauffman (2024).

ELEMENTARY BRAIDS



σ_i



σ_i^{-1}

e

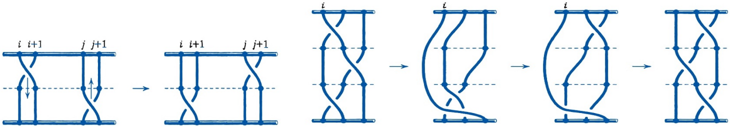
ARTIN BRAID GROUP

Artin(1925)

The braid group B_n, $n \geq 2$, on n strings can be defined as a group generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ with the defining relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j| \geq 2.$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad i = 1, 2, \ldots, n-2,$$



Braid group relations

REPRESENTATION OF B_n BY AUTOMORPHISMS OF FREE GROUP

$$\phi_A \colon B_n \longrightarrow \mathrm{Aut}(F_n),$$

where $F_n = \langle x_1, x_2, \ldots, x_n \rangle$ is a free group, is defined by the rule

$$\phi_A(\sigma_i) : \begin{cases} x_i \longmapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} \longmapsto x_i. \end{cases}$$

Here and onward we point out only nontrivial actions on generators assuming that other generators are fixed.

Theorem (Artin, 1925): $Ker(\phi_A) = 1$.

PURE BRAID GROUP

There is a homomorphism $\varphi : B_n \to S_n$, $\varphi(\sigma_i) = (i, i+1)$, $i = 1, 2, \ldots, n-1$. Its kernel $Ker(\varphi)$ is called the *pure braid group* and denoted by P_n.

The group P_n is generated by a_{ij}, $1 \le i < j \le n$.
These generators can be expressed by the generators of B_n as follows

$$a_{i,i+1} = \sigma_i^2,$$
$$a_{ij} = \sigma_{j-1}\,\sigma_{j-2}\ldots\sigma_{i+1}\,\sigma_i^2\,\sigma_{i+1}^{-1}\ldots\sigma_{j-2}^{-1}\,\sigma_{j-1}^{-1}, \quad i+1 < j \le n.$$

PURE BRAID GROUP

The generators of B_n act on the generator of P_n by the rules:

$$\sigma_k^{-1} a_{ij} \sigma_k = a_{ij}, \text{ where } k \neq i-1, i, j-1, j,$$

$$\sigma_i^{-1} a_{i,i+1} \sigma_i = a_{i,i+1},$$

$$\sigma_{i-1}^{-1} a_{ij} \sigma_{i-1} = a_{i-1,j},$$

$$\sigma_i^{-1} a_{ij} \sigma_i = a_{i+1,j} \, [a_{i,i+1}^{-1}, a_{ij}^{-1}], j \neq i+1.$$

$$\sigma_{j-1}^{-1} a_{ij} \, \sigma_{j-1} = a_{i,j-1},$$

$$\sigma_j^{-1} a_{ij} \sigma_j = a_{ij} a_{i,j+1} a_{ij}^{-1}, \text{ where}$$

$$[a, b] = a^{-1} b^{-1} ab = a^{-1} a^b,$$

PURE BRAID GROUP

$U_i = \langle a_{1i}, a_{2i}, \ldots, a_{i-1,i} \rangle, \quad i = 2, \ldots, n$, a subgroup of P_n.
It is known that U_i is a free group of rank $i-1$. P_n is defined by the following conjugation rules (for $\varepsilon = \pm 1$):

$$a_{ik}^{-\varepsilon} a_{kj} a_{ik}^{\varepsilon} = (a_{ij} a_{kj})^{\varepsilon} a_{kj} (a_{ij} a_{kj})^{-\varepsilon},$$

$$a_{km}^{-\varepsilon} a_{kj} a_{km}^{\varepsilon} = (a_{kj} a_{mj})^{\varepsilon} a_{kj} (a_{kj} a_{mj})^{-\varepsilon}, \ m < j,$$

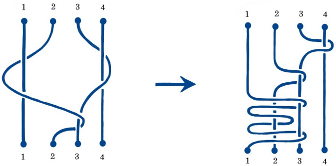
$$a_{im}^{-\varepsilon} a_{kj} a_{im}^{\varepsilon} = [a_{ij}^{-\varepsilon}, a_{mj}^{-\varepsilon}]^{\varepsilon} a_{kj} [a_{ij}^{-\varepsilon}, a_{mj}^{-\varepsilon}]^{-\varepsilon}, \ i < k < m.$$

$$a_{im}^{-\varepsilon} a_{kj} a_{im}^{\varepsilon} = a_{kj}, \ k < i < m < j \text{ or } m < k.$$

A. Markov (1945) proved that P_n is a semi-direct product of free groups:

$$P_n = U_n \rtimes (U_{n-1} \rtimes (\ldots \rtimes (U_3 \rtimes U_2)) \ldots),$$

$$U_i \simeq F_{i-1}, \quad i = 2, 3, \ldots, n.$$



VIRTUAL BRAID GROUP

L. Kauffman (1996)

Virtual braid group VB_n is generated by the classical braid group $B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ and the symmetric group $S_n = \langle \rho_1, \ldots, \rho_{n-1} \rangle$. Generators ρ_i satisfy the following relations:

$$\rho_i^2 = 1 \qquad \text{for } i = 1, 2, \ldots, n-1,$$
$$\rho_i \rho_j = \rho_j \rho_i \qquad \text{for } |i-j| \geq 2,$$
$$\rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1} \qquad \text{for } i = 1, 2 \ldots, n-2.$$

Also we have mixed defining relations

$$\sigma_i \rho_j = \rho_j \sigma_i \qquad \text{for } |i-j| \geq 2,$$
$$\rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1} \qquad \text{for } i = 1, 2, \ldots, n-2.$$

MULTIPLE VIRTUAL BRAID GROUP (GENERATORS)

Let k be a non-negative integer.

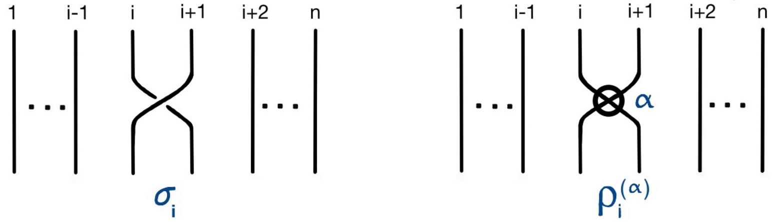
Multiple k-virtual braid group $kMVB_n$ presented by L. Kauffman (2024).

L. Kauffman, *Multi-Virtual Knot Theory*, https://arxiv.org/abs/2409.07499, 82 pp.

This group is generated by elements

$$\sigma_i, \quad \rho_i^{(\alpha)}, \quad i = 1, 2, \ldots, n-1, \ \alpha = 0, 1, \ldots, k-1.$$

GEOMETRIC INTERPRETATION OF GENERATORS



Geometric interpretation of σ_i and ρ_i^α

MULTIPLE VIRTUAL BRAID GROUP (RELATIONS), I

Defining relations of $kMVB_n$.

I. *Involutivity of generators*:

$$\left(\rho_i^{(\alpha)}\right)^2 = 1, \quad i = 1, 2, \ldots, n-1, \ \alpha = 0, 1, \ldots, k-1.$$

II. *Far commutativity*

– homogeneous:

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \ \rho_i^{(\alpha)} \rho_j^{(\alpha)} = \rho_j^{(\alpha)} \rho_i^{(\alpha)}, \quad |i-j| \geq 2;$$

– mixed ($\alpha \neq \beta$):

$$\sigma_i \rho_j^{(\alpha)} = \rho_j^{(\alpha)} \sigma_i, \ \rho_i^{(\alpha)} \rho_j^{(\beta)} = \rho_j^{(\beta)} \rho_i^{(\alpha)}, \quad |i-j| \geq 2.$$

MULTIPLE VIRTUAL BRAID GROUP (RELATIONS), II

III. *Braid relations*

– homogeneous:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},$$

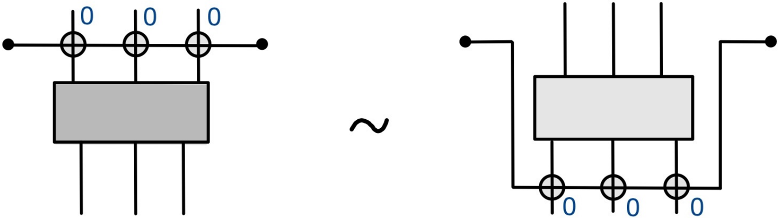
$$\rho_i^{(\alpha)} \rho_{i+1}^{(\alpha)} \rho_i^{(\alpha)} = \rho_{i+1}^{(\alpha)} \rho_i^{(\alpha)} \rho_{i+1}^{(\alpha)}, \quad i = 1, 2, \ldots, n-2;$$

– mixed ($\beta = 1, 2, \ldots, k-1$):

$$\sigma_i \rho_{i+1}^{(0)} \rho_i^{(0)} = \rho_{i+1}^{(0)} \rho_i^{(0)} \sigma_{i+1},$$

$$\rho_{i+1}^{(\beta)} \rho_i^{(0)} \rho_{i+1}^{(0)} = \rho_i^{(0)} \rho_{i+1}^{(0)} \rho_i^{(\beta)}, \qquad i = 1, 2, \ldots, n-2.$$

VIRTUAL DETOUR MOVE



Detour move

FORBIDDEN RELATIONS

Remark. Relations of the following forms are forbidden

$F1$: $\sigma_i \sigma_{i+1} \rho_i^{(\alpha)} = \rho_{i+1}^{(\alpha)} \sigma_i \sigma_{i+1}$, $0 \leq \alpha \leq k-1$;

$F2$: $\sigma_{i+1} \sigma_i \rho_{i+1}^{(\alpha)} = \rho_i^{(\alpha)} \sigma_{+1i} \sigma_i$, $0 \leq \alpha \leq k-1$;

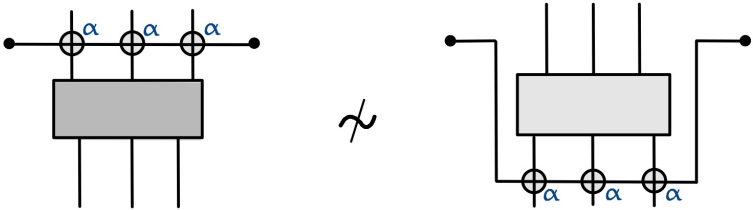
$F3$: $\rho_i^{(0)} \rho_{i+1}^{(\beta)} \rho_i^{(\beta)} = \rho_{i+1}^{(\beta)} \rho_i^{(\beta)} \rho_{i+1}^{(0)}, \quad \rho_i^{(\gamma)} \rho_{i+1}^{(\beta)} \rho_i^{(\beta)} = \rho_{i+1}^{(\beta)} \rho_i^{(\beta)} \rho_{i+1}^{(\gamma)},$

$\rho_i^{(\gamma)} \rho_{i+1}^{(\gamma)} \rho_i^{(\beta)} = \rho_{i+1}^{(\beta)} \rho_i^{(\gamma)} \rho_{i+1}^{(\gamma)}, \quad 0 < \gamma < \beta \leq k-1.$

If $k = 0$, then $kMVB_n$ is the braid group B_n.

If $k = 1$, then $kMVB_n$ is the virtual braid group VB_n.

FORBIDDEN DETOUR MOVE



Forbidden detour move ($\alpha > 0$)

SYMMETRIC MULTIPLE VIRTUAL BRAID GROUP

Also, L. Kauffman suggested to study analogous of $kMVB_n$ in which are allowed detour moves for any types of virtual crossings.

Let $k \geq 2$.
Symmetric multiple k-virtual braid group $\widetilde{kMVB}_n$ is a group which is quotient of $kMVB_n$ by the forbidden relations $F3$.

Remark. Forbidden relation in $\widetilde{kMVB}_n$:
$F1$: $\sigma_i \sigma_{i+1} \rho_i^{(\alpha)} = \rho_{i+1}^{(\alpha)} \sigma_i \sigma_{i+1}$,
$F2$: $\sigma_{i+1} \sigma_i \rho_{i+1}^{(\alpha)} = \rho_i^{(\alpha)} \sigma_{+1i} \sigma_i$.

FLAT VIRTUAL BRAID GROUP

L. Kauffman, S. Lambropoulou (2004) defined some quotients of VB_n. In particular, flat virtual braid group FVB_n.

FVB_n is the quotient of VB_n by relations

$$\sigma_i^2 = 1, \quad i = 1, 2, \ldots, n-1.$$

VIRTUAL PURE BRAID GROUP

Define an epimorphism $\varphi_n \colon VB_n \to S_n$ as follows:

$$\varphi_n(\sigma_i) = \rho_i, \ \varphi_n(\rho_i) = \rho_i, \ i = 1, 2, \ldots, n-1.$$

Its kernel $VP_n = Ker(\varphi_n)$ is the virtual pure braid group.

Define the following elements in VB_n:

$$\lambda_{i,i+1} = \rho_i \, \sigma_i^{-1}, \quad \lambda_{i+1,i} = \rho_i \, (\lambda_{i,i+1}) \, \rho_i = \sigma_i^{-1} \, \rho_i, \quad i = 1, 2, \ldots, n-1,$$

$$\lambda_{ij} = \rho_{j-1} \, \rho_{j-2} \ldots \rho_{i+1} \, (\lambda_{i,i+1}) \, \rho_{i+1} \ldots \rho_{j-2} \, \rho_{j-1},$$

$$\lambda_{ji} = \rho_{j-1} \, \rho_{j-2} \ldots \rho_{i+1} \, (\lambda_{i+1,i}) \, \rho_{i+1} \ldots \rho_{j-2} \, \rho_{j-1},$$

$$1 \leq i < j-1 \leq n-1.$$

DEFINING RELATIONS OF VP_n

V. Bardakov (2004)

The group VP_n ($n \geq 2$) admits a presentation with the generators $\lambda_{ij},\ 1 \leq i \neq j \leq n$, and the following relations:

$$\lambda_{ij}\lambda_{kl} = \lambda_{kl}\lambda_{ij},$$
$$\lambda_{ki}\lambda_{kj}\lambda_{ij} = \lambda_{ij}\lambda_{kj}\lambda_{ki},$$

where distinct letters stand for distinct indices.

ANOTHER EPIMORPHISM OF VB_n TO S_n

L. Rabenda (2003) defined another epimorphism $\psi_n \colon VB_n \to S_n$ as follows:

$$\psi_n(\sigma_i) = 1,\ \psi_n(\rho_i) = \rho_i,\ i = 1, 2, \ldots, n-1.$$

We will call its kernel $H_n = Ker(\psi_n)$ by virtual semi-pure braid group.

L. Rabenda proved that H_n is generated by elements $x_{i,j}$, $1 \leq i \neq j \leq n$, where

$$x_{i,i+1} = \sigma_i, \quad x_{i+1,i} = \rho_i \sigma_i \rho_i = \rho_i x_{i,i+1} \rho_i, i = 1, 2, \ldots, n-1,$$

$$x_{i,j} = \rho_{j-1} \cdots \rho_{i+1}(\sigma_i)\rho_{i+1} \cdots \rho_{j-1},$$

$$x_{j,i} = \rho_{j-1} \cdots \rho_{i+1}(\rho_i \sigma_i \rho_i)\rho_{i+1} \cdots \rho_{j-1}, 1 \leq i < j-1 \leq n-1.$$

DEFINING RELATIONS OF H_n

H_n is defined by relations:

$$x_{i,j}\, x_{k,l} = x_{k,l}\, x_{i,j},$$

$$x_{i,k}\, x_{k,j}\, x_{i,k} = x_{k,j}\, x_{i,k}\, x_{k,j},$$

where distinct letters stand for distinct indices.

L. Rabenda (2003)

$$VB_n = H_n \rtimes S_n,$$

where $S_n = \langle \rho_1, \ldots, \rho_{n-1} \rangle$ acts on the set by permutation of indices.

MULTIPLE VIRTUAL PURE BRAID GROUP

For simplicity we shall denote $\rho_i = \rho_i^{(0)}$.

We know that a group which is generated by ρ_1, ρ_2, ..., ρ_{n-1} is isomorphic to the symmetric group S_n.

Define a map on the generators

$$\varphi_{n,k} \colon kMVB_n \to S_n, \quad \sigma_i \mapsto \rho_i, \quad \rho_i^{(\alpha)} \mapsto \rho_i,$$

$$i = 1, 2, \ldots, n-1, \; \alpha = 1, 2, \ldots, k-1.$$

This map induces an epimorphism and the kernel $\ker \varphi_{n,k}$ is the multiple virtual pure braid group $kMVP_n$.

MULTIPLE VIRTUAL SEMI-PURE BRAID GROUP

Also, we can define a second map on the generators

$$\psi_{n,k} \colon kMVB_n \to S_n, \quad \sigma_i \mapsto 1, \quad \rho_i^{(\alpha)} \mapsto \rho_i,$$

$$i = 1, 2, \ldots, n-2, \ \alpha = 1, 2, \ldots, k-1.$$

This map induces an epimorphism and the kernel $\ker \psi_{n,k}$ is the multiple virtual semi-pure braid group $kMVH_n$.

GENERATORS OF $kMVP_n$

Let us define the following elements in $kMVB_n$:

$$\lambda_{i,i+1}^{(0)} = \rho_i \, \sigma_i^{-1},$$
$$\lambda_{i+1,i}^{(0)} = \rho_i \, \lambda_{i,i+1}^{(0)} \, \rho_i = \sigma_i^{-1} \, \rho_i,$$
$$\lambda_{i,i+1}^{(\beta)} = \rho_i \, \rho_i^{(\beta)}, \quad 1 \leq \beta \leq k-1,$$

for $i = 1, 2, \ldots, n-1$, and

$$\lambda_{i,j}^{(0)} = \rho_{j-1} \, \rho_{j-2} \ldots \rho_{i+1} \, \lambda_{i,i+1}^{(0)} \, \rho_{i+1} \ldots \rho_{j-2} \, \rho_{j-1},$$
$$\lambda_{j,i}^{(0)} = \rho_{j-1} \, \rho_{j-2} \ldots \rho_{i+1} \, \lambda_{i+1,i}^{(0)} \, \rho_{i+1} \ldots \rho_{j-2} \, \rho_{j-1},$$
$$\lambda_{i,j}^{(\beta)} = \rho_{j-1} \, \rho_{j-2} \ldots \rho_{i+1} \, \lambda_{i,i+1}^{(\beta)} \, \rho_{i+1} \ldots \rho_{j-2} \, \rho_{j-1},$$

for $1 \leq i < j-1 \leq n-1, \quad 0 < \beta \leq k-1.$

MULTIPLE VIRTUAL PURE BRAID GROUP

THEOREM 1. [V.Bardakov, T.K., K.Negi, M.Prabhakar 2025]

$kMVP_n$ admits a presentation with the generators

$$\lambda_{ij}^{(\alpha)}, \quad \lambda_{ji}^{(0)}, \quad 1 \leq i < j \leq n, \quad 0 \leq \alpha \leq k-1,$$

and the following relations:

$$\lambda_{ij}^{(\alpha)} \lambda_{kl}^{(\beta)} = \lambda_{kl}^{(\beta)} \lambda_{ij}^{(\alpha)}, \quad 0 \leq \alpha \leq \beta \leq k-1,$$

$$\lambda_{ik}^{(0)} \lambda_{jk}^{(0)} \lambda_{ij}^{(0)} = \lambda_{ij}^{(0)} \lambda_{jk}^{(0)} \lambda_{ik}^{(0)}, \quad 1 \leq i, j, k \leq n,$$

$$\lambda_{ik}^{(\alpha)} \lambda_{jk}^{(\alpha)} \lambda_{ij}^{(\alpha)} = \lambda_{ij}^{(\alpha)} \lambda_{jk}^{(\alpha)} \lambda_{ik}^{(\alpha)}, \quad 1 \leq i < j < k \leq n, \quad 1 \leq \alpha \leq k-1,$$

where distinct letters stand for distinct indices.

STRUCTURE OF $kMVB_n$

COROLLARY 1. [V. Bardakov, T.K., K. Negi, M. Prabhakar]

1. $kMVB_n = kMVP_n \rtimes S_n$.
2. $\langle \lambda_{ij}^{(0)} \mid 1 \leq i \neq j \leq n \rangle \cong VP_n$.
3. For any $\alpha \in \{1, 2, \ldots, k-1\}$ holds $\langle \lambda_{ij}^{(\alpha)} \mid 1 \leq i \neq j \leq n \rangle \cong FVP_n$.
4. If $n = 3$, then $kMVP_3 \cong VP_3 * (FVP_3)^{*(k-1)}$.

ACTION OF S_n ONTO $kMVP_n$

We have decomposition $kMVB_n = kMVP_n \rtimes S_n$ and S_n acts on $kMVP_n$ by the rules

PROPOSITION 1.

Let a be an element of $\langle \rho_1, \rho_2, \ldots, \rho_{n-1} \rangle$ and $\bar{a}$ is its image in S_n under the isomorphism $\rho_i \mapsto (i, i+1)$, $i = 1, 2, \ldots, n-1$, then for any generator $\lambda_{ij}^{(\alpha)}$ of $kMVP_n$ the following holds

$$a^{-1} \lambda_{ij}^{(\alpha)} a = \lambda_{(i)\bar{a},(j)\bar{a}}^{(\alpha)},$$

where $(k)\bar{a}$ is the image of k under the action of the permutation $\bar{a}$.

GENERATORS OF $kMVH_n$

Let us define the following elements in $kMVH_n$:

$$x_{i,i+1}^{(0)} = \sigma_i, \quad x_{i+1,i}^{(0)} = \rho_i \, x_{i,i+1}^{(0)} \, \rho_i,$$

$$x_{i,i+1}^{(\beta)} = \rho_i \, \rho_i^{(\beta)}, \quad 1 \leq \beta \leq k-1,$$

for $i = 1, 2, \ldots, n-1$, and

$$x_{i,j}^{(\alpha)} = \rho_{j-1} \, \rho_{j-2} \ldots \rho_{i+1} \, x_{i,i+1}^{(\alpha)} \, \rho_{i+1} \ldots \rho_{j-2} \, \rho_{j-1},$$

$$x_{j,i}^{(0)} = \rho_{j-1} \, \rho_{j-2} \ldots \rho_{i+1} \, x_{i+1,i}^{(0)} \, \rho_{i+1} \ldots \rho_{j-2} \, \rho_{j-1},$$

for $1 \leq i < j-1 \leq n-1, \quad 0 \leq \alpha \leq k-1.$

MULTIPLE VIRTUAL SEMI-PURE BRAID GROUP

THEOREM 2. [V.Bardakov, T.K., K.Negi, M.Prabhakar 2025]
$kMVH_n$ admits a presentation with the generators

$$x_{ij}^{(0)}, \quad 1 \leq i \neq j \leq n,$$

$$x_{ij}^{(\beta)}, \quad 1 \leq i < j \leq n, \quad 1 \leq \beta \leq k-1,$$

and the following relations:

$$x_{ij}^{(\alpha)} x_{kl}^{(\gamma)} = x_{kl}^{(\gamma)} x_{ij}^{(\alpha)}, \quad 0 \leq \alpha \leq \gamma \leq k-1,$$
$$x_{ik}^{(0)} x_{kj}^{(0)} x_{ik}^{(0)} = x_{kj}^{(0)} x_{ik}^{(0)} x_{kj}^{(0)}, \quad 1 \leq i, j, k \leq n,$$
$$x_{ik}^{(\beta)} x_{jk}^{(\beta)} x_{ij}^{(\beta)} = x_{ij}^{(\beta)} x_{jk}^{(\beta)} x_{ik}^{(\beta)}, \quad 1 \leq i < j < k \leq n, \quad 1 \leq \beta \leq k-1,$$

where distinct letters stand for distinct indices.

PROPERTIES OF $kMVH_n$

COROLLARY 2. [V. Bardakov, T.K., K. Negi, M. Prabhakar]

1. $kMVB_n = kMVH_n \rtimes S_n$.
2. $\langle x_{ij}^{(0)} \mid 1 \leq i \neq j \leq n \rangle \cong H_n$.
3. For any $\beta \in \{1, 2, \ldots, k-1\}$ holds $\langle x_{ij}^{(\beta)} \mid 1 \leq i \neq j \leq n \rangle \cong FH_n$.
4. If $n = 3$, then $kMVH_3 \cong H_3 * (FH_3)^{*(k-1)}$.

ACTION BY S_n

We have decomposition $kMVB_n = kMVH_n \rtimes S_n$ and S_n acts on $kMVH_n$ by the rules

LEMMA 1.

Let a be an element of $\langle \rho_1, \rho_2, \ldots, \rho_{n-1} \rangle$ and $\bar{a}$ is its image in S_n under the isomorphism $\rho_i \mapsto (i, i+1)$, $i = 1, 2, \ldots, n-1$, then for any generator $x_{ij}^{(\alpha)}$ of $kMVH_n$ the following holds

$$a^{-1} x_{ij}^{(\alpha)} a = x_{(i)\bar{a},(j)\bar{a}}^{(\alpha)},$$

where $(k)\bar{a}$ is the image of k under the action of the permutation $\bar{a}$ and we use the relations

$$x_{ji}^{(\beta)} = \left(x_{ij}^{(\beta)} \right)^{-1}, \quad 1 \leq \beta \leq k-1.$$

KNOWN REPRESENTATIONS OF VB_n

V. Vershinin defined the representation $\varphi_V \colon VB_n \longrightarrow \mathrm{Aut}(F_n)$:

$$\varphi_V(\sigma_i)\colon \begin{cases} x_i \longmapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} \longmapsto x_i, \end{cases} \qquad \varphi_V(\rho_i)\colon \begin{cases} x_i \longmapsto x_{i+1}, \\ x_{i+1} \longmapsto x_i. \end{cases}$$

This representation has the nontrivial kernel for every $n > 2$.

V. Manturov and independently V. Bardakov defined the representation $\varphi_{MB} \colon VB_n \longrightarrow \mathrm{Aut}(F_{n+1})$, $F_{n+1} = \langle x_1, x_2, \ldots, x_n, y \rangle$:

$$\varphi_{MB}(\sigma_i)\colon \begin{cases} x_i \longmapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} \longmapsto x_i, \end{cases} \qquad \varphi_{MB}(\rho_i)\colon \begin{cases} x_i \longmapsto x_{i+1}^{y^{-1}}, \\ x_{i+1} \longmapsto x_i^{y}. \end{cases}$$

KNOWN REPRESENTATIONS OF VB_n

D. Silver and S. Williams defined the representation
$\varphi_{SW}\colon VB_n \longrightarrow \mathrm{Aut}(F_{n,n+1})$, where
$F_{n,n+1} = F_n * \mathbb{Z}^{n+1} = \langle x_1, x_2, \ldots, x_n, v, u_1, u_2, \ldots, u_n \rangle$:

$$\varphi_{SW}(\sigma_i)\colon \begin{cases} x_i \longmapsto x_i x_{i+1}^{u_i} x_i^{-vu_{i+1}}, \\ x_{i+1} \longmapsto x_i^{v}, \end{cases} \qquad \varphi_{SW}(\sigma_i)\colon \begin{cases} u_i \longmapsto u_{i+1}, \\ u_{i+1} \longmapsto u_i, \end{cases}$$

$$\varphi_{SW}(\rho_i)\colon \begin{cases} x_i \longmapsto x_{i+1}, \\ x_{i+1} \longmapsto x_i, \end{cases} \qquad \varphi_{SW}(\rho_i) : \begin{cases} u_i \longmapsto u_{i+1}, \\ u_{i+1} \longmapsto u_i. \end{cases}$$

Boden - Dies - Gaudreau - Gerlings - Harper - Nicas defined the representation $\varphi_{BD}\colon VB_n \longrightarrow \mathrm{Aut}(F_{n,2})$:

$$\varphi_{BD}(\sigma_i)\colon \begin{cases} x_i \longmapsto x_i x_{i+1} x_i^{-u}, \\ x_{i+1} \longmapsto x_i^{u}, \end{cases} \qquad \varphi_{BD}(\rho_i)\colon \begin{cases} x_i \longmapsto x_{i+1}^{v^{-1}}, \\ x_{i+1} \longmapsto x_i^{v}. \end{cases}$$

GENERALISATION OF THE PREVIOUS REPRESENTATIONS

V. Bardakov, Yu. Mikhalchishina, M. Neshchadim (2017)

The representation $\varphi_M \colon VB_n \longrightarrow \mathrm{Aut}(F_{n,n})$ defined by the action on the generators

$$\varphi_M(\sigma_i)\colon \begin{cases} x_i \longmapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} \longmapsto x_i, \end{cases} \qquad \varphi_M(\sigma_i)\colon \begin{cases} v_i \longmapsto v_{i+1}, \\ v_{i+1} \longmapsto v_i, \end{cases}$$

$$\varphi_M(\rho_i)\colon \begin{cases} x_i \longmapsto x_{i+1}^{v_i^{-1}}, \\ x_{i+1} \longmapsto x_i^{v_{i+1}}, \end{cases} \qquad \varphi_M(\rho_i)\colon \begin{cases} v_i \longmapsto v_{i+1}, \\ v_{i+1} \longmapsto v_i \end{cases}$$

Any of representations φ_V, φ_{MB}, φ_{SW}, φ_{BD} can be get from φ_M.

NONTRIVIAL KERNELS OF KNOWN REPRESENTATIONS

O. Chterental proved that the representation φ_{MB} of the group VB_n is not faithful for every $n \geq 4$.

O. Chterental (2015)

The element $b = (\sigma_2^{-1}\rho_1\sigma_2\rho_3)^3 \in VB_4$ lies in the kernel of the representations φ_{MB}.

V. Bardakov, Yu. Mikhalchishina, M. Neshchadim (2017)

For every $n \geq 4$ the element b lies in the kernels of φ_{SW} and φ_{BD}, but does not lies in the kernel of φ_M.

REPRESENTATION OF $kMVB_n$

Let $F_{n,kn} = F_n * \mathbb{Z}^{kn}$, where $F_n = \langle x_1, x_2, \ldots, x_n \rangle$ is the free group and $\mathbb{Z}^{kn} = \langle a_1^{(\alpha)}, a_2^{(\alpha)}, \ldots, a_n^{(\alpha)} \rangle$, is the free abelian group.
Put

$$\Phi_k(\sigma_i)\colon \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} \mapsto x_i, \end{cases} \qquad \Phi_k(\sigma_i)\colon \begin{cases} a_i^{(\alpha)} \mapsto a_{i+1}^{(\alpha)}, \\ a_{i+1}^{(\alpha)} \mapsto a_i^{(\alpha)}, \end{cases}$$

$$\Phi_k(\rho_i^{(\alpha)})\colon \begin{cases} x_i \mapsto a_i^{(\alpha)} x_{i+1} \left(a_i^{(\alpha)}\right)^{-1}, \\ \\ x_{i+1} \mapsto \left(a_{i+1}^{(\alpha)}\right)^{-1} x_i \, a_{i+1}^{(\alpha)}, \end{cases} \qquad \Phi_k(\rho_i^{(\alpha)})\colon \begin{cases} a_i^{(\alpha)} \mapsto a_{i+1}^{(\alpha)}, \\ a_{i+1}^{(\alpha)} \mapsto a_i^{(\alpha)}, \end{cases}$$

for all $i = 1, 2, \ldots, i-1$, $\alpha = 0, 1, \ldots, k-1$.

REPRESENTATION OF $kMVB_n$

This representation is an extension of the Artin representation $B_n \to Aut(F_n)$.

THEOREM 3. [V.Bardakov, T.K., 2025]

The map Φ_k, $k = 1, 2, \ldots$, defines representations of $kMVB_n$ and $\widetilde{kMVB}_n$ into $Aut(F_{n,kn})$.

COROLLARY 3. [V. Bardakov, T.K., 2025]

The representation $\Phi_k : kMVB_n \to Aut(F_{n,kn})$ is not faithful for all $n > 2$ and $k > 1$.

FAITHFULNESS

It is not difficult to see that for $n = 2$ and $k > 1$ the group $kMVB_2$ is isomorphic to $\widetilde{kMVB}_2$.

QUESTION 1.

For which values $n > 2$ and $k > 1$ the representation

$$\Phi_k\colon \widetilde{kMVB}_n \to Aut(F_{n,kn})$$

is faithful?

GROUP OF LINK

Let $\mathcal{L}$ be the set of all links in $\mathbb{R}^3$.

A group $G(L)$ of a link $L \in \mathcal{L}$ is a group $\pi_1(\mathbb{R}^3 \setminus L)$.

Artin, (1925)

If L is isotopic to $\hat{\beta}$, where $\beta \in B_n$, then

$$G(L) = \langle x_1, x_2, \ldots, x_n \parallel x_i = \varphi_A(\beta)(x_i), \quad i = 1, 2, \ldots, n \rangle.$$

GROUPS OF VIRTUAL LINK

Assume that we have a representation $\psi\colon VB_n \longrightarrow \mathrm{Aut}(H)$ of the virtual braid group into the automorphism group of some group

$$H = \langle h_1, h_2, \ldots, h_m \parallel \mathcal{R} \rangle,$$

where $\mathcal{R}$ is the set of defining relations.

The following group is assigned to the virtual braid $\beta \in VB_n$:

$$G_\psi(\beta) = \langle h_1, h_2, \ldots, h_m \parallel \mathcal{R}, h_i = \psi(\beta)(h_i), \quad i = 1, 2, \ldots, m \rangle.$$

The group G_ψ is an invariant of virtual links if the group $G_\psi(\beta)$ is isomorphic to $G_\psi(\beta')$ for each braid β' such that the links $\widehat{\beta}$ and $\widehat{\beta}'$ are equivalent.

GROUPS OF VIRTUAL LINK

Using the representation $\varphi_M \colon VB_n \longrightarrow \mathrm{Aut}(F_{n,n})$ we can define a group of virtual link.

V. Bardakov, Yu. Mikhalchishina, M. Neshchadim (2017)

Given $\beta \in VB_n$ and $\beta' \in VB_m$ the two virtual braids such that theirs closures define the same link L, then $G_M(\beta) \cong G_M(\beta')$.

TOPOLOGICAL APPROACH TO GROUPS OF VIRTUAL LINK

In the paper:

J. S. Carter, D. Silver, S. Williams, *Invariants of links in thickened surfaces*, Algebr. Geom. Topol., **14** (2014), no. 3, 1377–1394.

suggested another approach to the definition of virtual link groups, which used interpretation of virtual link as a classical link in a thickened surface.

KNOT INVARIANT OF MULTIPLE VIRTUAL LINKS

Assume also, that we have a representation

$$\Phi \colon kMVB_n \longrightarrow \mathrm{Aut}(H).$$

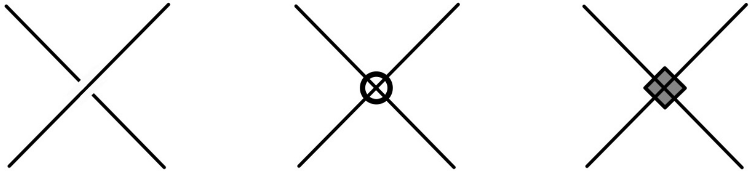
For a multiple virtual braid $\beta \in kMVB_n$ define the following algebraic system:

$$A_\Phi(\beta) = \langle h_1, h_2, \ldots, h_m \parallel \mathcal{R}, h_i = \Phi(\beta)(h_i), \quad i = 1, 2, \ldots, m \rangle.$$

This algebraic system A_Φ is an invariant of multiple virtual links if $A_\Phi(\beta)$ is isomorphic to $A_\Phi(\beta')$ for each braid β' such that the links $\widehat{\beta}$ and $\widehat{\beta'}$ are equivalent.

2-VIRTUAL LINKS AND THEIR INVARIANTS

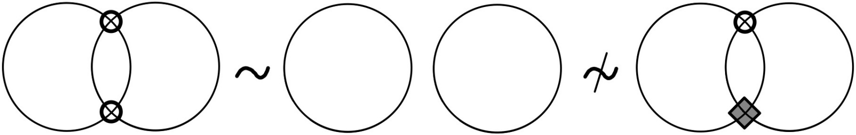
2-virtual link diagram is a diagram on the plane with three types of crossings: classical and two types of virtual crossings:



3 types of crossings

2-virtual link is equivalent class of 2-virtual link diagrams by generalized Reidemeister moves.

TWO TYPES OF HOPF LINKS



Trivial link and Hopf link with different crossings

KNOTS WITH ONLY VIRTUAL CROSSINGS

Hopf link with two virtual crossings is not equivalent to a disjoint union of two circles.

L. Kauffman suggested the following question (see [Kauf-24, p. 39]).

L. Kauffman, *Multi-Virtual Knot Theory*, https://arxiv.org/abs/2409.07499, 82 pp.

QUESTION

Is it true that any single component diagram, decorated only with two virtual crossings, reduces by detour moves to a circle.

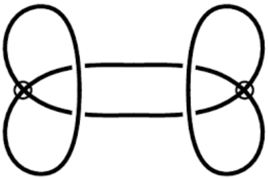
KNOTS WITH ONLY VIRTUAL CROSSINGS

THEOREM 4. [V.Bardakov, T.K., 2025]

There are 2-virtual knots with only virtual crossings which are not equivalent to the trivial knot.

KISHINO KNOT

The Kishino knot is a non-trivial knot that is the connected sum of two trivial knots.

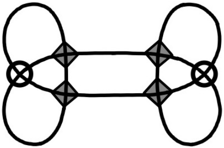


Kishino knot

There are different proofs that it is non-trivial.

FLAT KISHINO KNOT

If we take flat crossings instead of the classical crossings, we get the flat Kishino which also is non-trivial.



Flat Kishino knot

FLAT KISHINO KNOT

L. Kauffman and V. Manturov (2005) proved that that the flat Kishino knot is non-trivial.

They constructed a virtual biquandle for flat Kishino which is different from virtual biquandle of the trivial knots.

By this biquandle we can define a free module M over some rings and two linear maps

$$S, V \colon M \times M \to M \times M.$$

These maps give a representation of the flat virtual braid group

$$\Psi \colon FVB_n \to Aut(M)$$

REPRESENTATION OF $2MVB_n$

Using

$$\Psi\colon FVB_n \to Aut(M),$$

we can construct virtual biquandle $BQ_\Psi(\beta)$ for any $\beta \in FVB_n$, which is invariant of the flat virtual link $\hat{\beta}$.

Since there is a homomorphism

$$2MVB_n \to FVB_n, \quad \sigma_i \mapsto \tau_i, \quad \rho_i \mapsto \rho_i, \quad \tau_i \mapsto \tau_i,$$

the virtual biquandle $BQ_\Psi(\beta)$ is, in fact an invariant of 2-virtual knots.

CLASSICAL, FLAT AND MULTI VIRTUAL KNOTS

Kauffman (2024) proved that the classical knot theory embeds in the multiple virtual knot theory.

QUESTION

Is it true that the virtual knot theory (flat virtual knot theory) embeds in the multiple virtual knot theory?

It is true for the corresponding braid groups.

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