The number of prime knots and growth rates in (semi)groups

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Groups and quandles in low-dimensional topology

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Our central object

Our central object is the sequence

$$P_1, \quad P_2, \quad P_3, \quad P_4, \quad P_5, \quad \ldots$$

where P_n is the number of prime knots of n crossings.

Our central problem is to describe the asymptotic behavior of P_n.
(Conjecturally, we have $P_n \sim (7 \pm \varepsilon)^n$.)

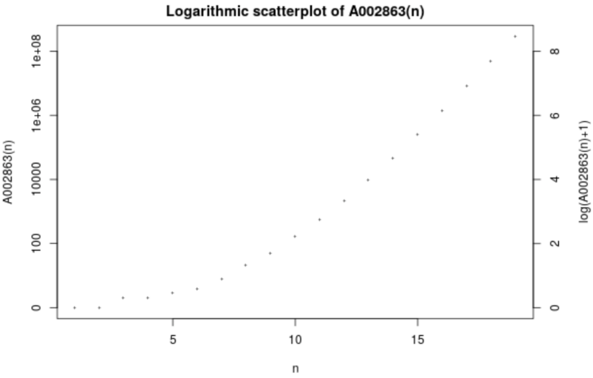
This problem can be regarded as the knot-theory analogue of the classical problem of the asymptotic distribution of prime numbers.

Our central object

n	P_n
1	0
2	0
3	1
4	1
5	2
6	3
7	7
8	21
9	49
10	165
11	552
12	2 176
13	9 988
14	46 972
15	253 293
16	1 388 705
17	8 053 393
18	48 266 466
19	294 130 458

* B. Burton

Our sequence $(P_n)_{n\in\mathbb{N}}$ is A002863 in The Sloane On-Line Encyclopedia of Integer Sequences (OEIS `http://oeis.org/`)



Why is the asymptotic behavior of P_n of interest?

1.

Conjecture (Thurston–Adams)

The proportion of hyperbolic knots amongst all of the prime knots of n or fewer crossings approaches 1 as n approaches infinity.

We proved that

$$\frac{\sum_{i=1}^{n} H_i}{\sum_{i=1}^{n} P_i} \not\to 1. \tag{1}$$

However, this does not imply even that

$$\limsup \frac{\sum_{i=1}^{n} H_i}{\sum_{i=1}^{n} P_i} \neq 1. \tag{2}$$

While we want to prove

$$\lim \frac{\sum_{i=1}^{n} H_i}{\sum_{i=1}^{n} P_i} = \lim \frac{H_n}{P_n} = 0. \tag{3}$$

Knowing the asymptotics of P_n gives an alternative way for proving (1). We hope that it could help proving (2) and (perhaps) (3).

Why is the asymptotic behavior of P_n of interest?

2.

The problem (of the asymptotic of P_n) is related to classical combinatorial problems:

- Asymptotic distribution of the prime numbers (recall $\pi(N) \sim \frac{N}{\ln(N)}$).
- The asymptotics of the number of *meanders*.
  (The number of all meander multicurves of order n is just the square of the nth Catalan number $C_n = \frac{(2n)!}{n!(n+1)!}$.)
- The asymptotic number of unicursal plane curves (recall Tutte's "census of planar maps"), etc.

This is the class of problems united by a common "mysterious" phenomenon: the total number of objects is (relatively) easily calculated, and a subset of objects with an additional property (for example, one-component ones) gives a rather difficult task.

Why is the asymptotic behavior of P_n of interest?

3.

The study of asymptotics of P_n forces us to find new regular (algebraic) structures in the (seemingly formless) set of knots and links.

This is related, in particular, to the study of metrics and Gordian graphs on the “space” of knot and links.

For example, the search for new bounds of the asymptotics of P_n made it possible to discover regular embeddings of some groups and semigroups into the space of knots (details ahead).

Main result: A new estimate

Theorem (a new estimate; joint result with Artem Aleshin)

Let P_n denote the number of prime knots of n crossings. Then

$$4.52 \quad < \quad \liminf_{n\to\infty} \sqrt[n]{P_n} \quad \leq \quad \limsup_{n\to\infty} \sqrt[n]{P_n} \quad < \quad 10.4.$$

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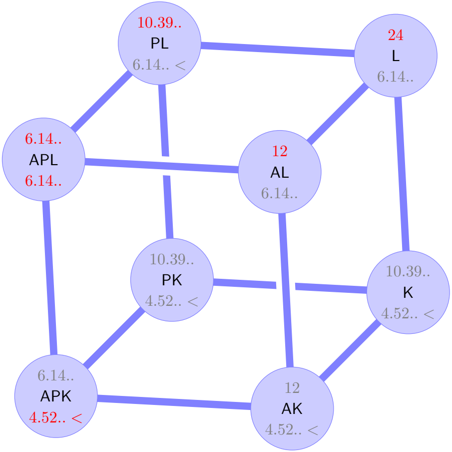
Here, "<10.4" follows from results of Stoimenow (2004) based on Sundberg and Thistlethwaite (1998).
Previously known lower bounds are:

- $4.45<$ (2021).
- $4<$ Alekseev, Malyutin, Vershik.
- $2.68<$ a widely known misprint by Welsh (1992).
- $2.13<$ Ernst and Sumners (1987).

The next aim is to show "$4.765<$" (in order to have a lower bound x such that $x^3 > (10.4)^2$).

Upper and lower bounds for related sequences



‘K’ stands for ‘knots’
‘L’ stands for ‘links’
‘P’ stands for ‘prime’
‘A’ stands for ‘alternating’; e. g., ‘APK’ means ‘alternating prime knots’

Description of the new method

Building blocks of the new method:

- Tait's conjectures (which give us a crossing-number-preserving injection of the set $\mathscr{C}$ of flype-equivalence classes of prime closed spherical curves into the set of [alternating] prime knots).
- Injections of semigroups of so-called (*colored weighted projective*) *heaps* to $\mathscr{C}$.
- Estimates on the growth rate for semigroups of (colored weighted projective) heaps.[1]

The new method has several versions:

(L) The "light" version: it uses usual simplest heaps (locally free semigroups). This version gives $4 \leq \liminf_{n\to\infty} \sqrt[n]{P_n}$.

(U) "Upgraded" versions, which use colored weighted heaps. With weights up to 6, one of these versions gives $4.45 < \liminf_{n\to\infty} \sqrt[n]{P_n}$, With weights up to 12, we obtain $4.52 < \liminf_{n\to\infty} \sqrt[n]{P_n}$.

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[1] A. M. Vershik, S. Nechaev, R. Bikbov, "Statistical Properties of Locally Free Groups with Applications to Braid Groups and Growth of Random Heaps".

Usual heaps (for the light version of the new method)

Definition (locally free groups and semi-groups)

The *locally free* group (semi-group) $\mathcal{LF}_k$ ($\mathcal{LF}_k^+$) with k generators is determined by the following presentation:

$$\langle \sigma_1, \dots, \sigma_k \mid \sigma_i \sigma_j = \sigma_j \sigma_i,\ |i-j| \geq 2 \rangle.$$

Also known as
(see A. J. Duncan, I. V. Kazachkov, V. N. Remeslennikov, “Parabolic and quasiparabolic subgroups of free partially commutative groups”):

- semifree groups [1, 2],
- graph groups [14, 22, 24, 26, 31, 33],
- right-angled Artin groups [4, 5, 6, 8, 10, 23, 30, 35],
- trace groups [13, 29],
- locally free groups [9, 28, 34]
- (free) partially commutative groups [3, 7, 11, 12, 15, 17, 18, 21, 25, 27, 32].

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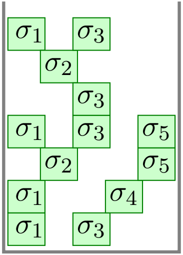

Graphical representation of the element

$$\sigma_1^2 \sigma_3 \sigma_4 \sigma_5^2 \sigma_2 \sigma_1 \sigma_3^2 \sigma_2 \sigma_1 \sigma_3$$

from $\mathcal{LF}_5^+$.

Usual heaps (for the light version of the new method)

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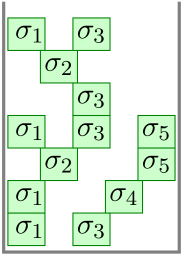
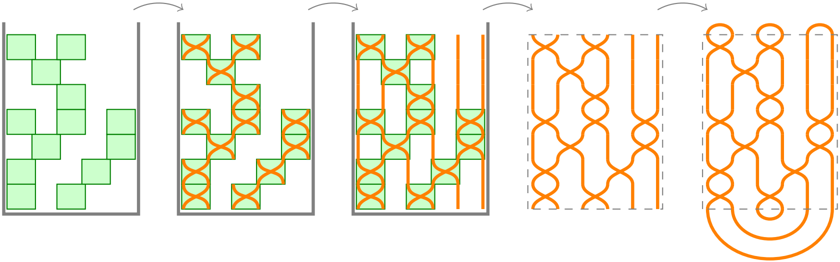


- Each pair of neighboring generators (σ_i, σ_{i+1}) produces a free subgroup (sub-semigroup).
- Each pair of non-neighboring generators (σ_i, σ_j), $|i-j| \geq 2$, produces a free abelian subgroup (sub-semigroup).
- The braid group B_{k-1} is a factor-group of $\mathcal{LF}_k$.

Heaps and curves

Turning a heap into a curve (the light version):



(1) We take the standard graphical presentation of a given heap in the form of a pile of "bricks" (in the tetris-style) in $\mathbb{R}^2$ (the left picture);

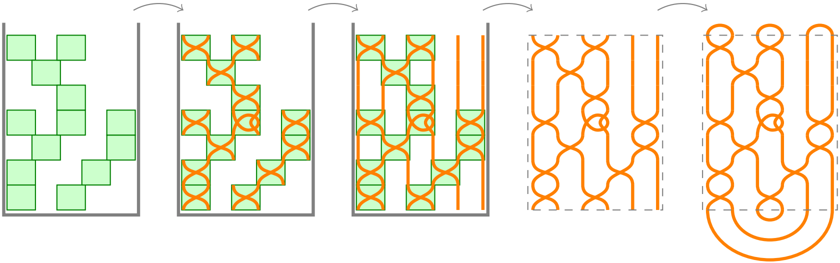
(2) Put a "crossing" in each brick of this pile (second left);

(3) Add vertical segments to obtain a picture similar to braid projection;

(4) Add some (perhaps, intersecting) arcs (outside of the dashed rectangle) in order to turn the obtained braid projection into a curve.

Enhanced versions of new method

Enhanced versions of new method use the same piles of “bricks”, but in addition to “crossings”, we allow to insert more complex fragments of curves into “bricks”.



In this figure, the fragment into the “central brick” is not a crossing. Recall that we consider curves and fragments of curves up to flype-equivalence.

Enhanced versions of new method

Semigroups of *colored weighted heaps*, which we use here, have three-parameter sets of generators of the form

$$\sigma_{x,y,z}$$

– the first index x shows generator's "position"
(similar to locally free groups);
– the second index y indicates the "weight" of the generator
(corresponds to the number of crossings);
– the third index z is a "color"
(we need these "colors" in order to distinguish between generators of the same weight, i. e., between non-flype-equivalent fragments of curves having the same number of crossings).

Enhanced versions of new method

Semigroups of *colored weighted heaps*, which we use here, have three-parameter sets of generators of the form

$$\sigma_{x,y,z}.$$

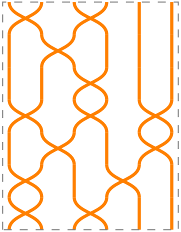
And the relations are

$$\sigma_{x,y,z}\sigma_{a,b,c} = \sigma_{a,b,c}\sigma_{x,y,z},\ |x-a| \geq 2,$$
$$\sigma_{x,y,z}\sigma_{x,1,1} = \sigma_{x,1,1}\sigma_{x,y,z},\ \forall x,y,z.$$

Note that for each x, we have a unique fragment with a single crossing, so that a unique generator of weight 1 appears, and we use a single color 1 for these weight-1 generators: $\sigma_{x,1,1}$. For brevity, we write simply σ_x instead of $\sigma_{x,1,1}$:

$$\sigma_x = \sigma_{x,1,1}.$$

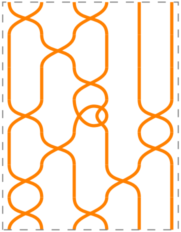
Comparing usual heaps and colored weighted ones



Graphical representation of the element

$$\sigma_1^2\sigma_3\sigma_4\sigma_5^2\sigma_2\sigma_1\sigma_3^2\sigma_2\sigma_1\sigma_3$$

from $\mathcal{L}\mathcal{F}_5^+$.



Graphical representation of the element

$$\sigma_1^2\sigma_3\sigma_4\sigma_5^2\sigma_2\sigma_1{\color{red}\sigma_{3,3,1}}\sigma_3\sigma_2\sigma_1\sigma_3.$$

A formula for the number of colored weighted heaps of total weight N

$$\sum_{m\in\{1,\dots,N\}} \Delta(m)\times \sum_{\substack{m_1;s_1,s_2,\dots s_p:\\ s_1+2s_2+3s_3+\dots+ps_p=N;\\ m_1\le m;\ m_1\le s_1}} A(m,m_1)\times B(m,m_1)\times C\times D$$

where

- $\Delta(m)\approx 3^m$ is the number of projective heaps (of a fixed large width) with m elements;
- $A(m,m_1)=C_m^{m_1}=\frac{m!}{m_1!(m-m_1)!}$ is the number of ways to choose m_1 block out of m ones (we suppose that precisely m_1 out of m blocks are composed of weight-1-elements only);
- $B(m,m_1)=C_{s_1-m_1+m-1}^{m-1}=\frac{(s_1-m_1+m-1)!}{(m-1)!(s_1-m_1)!}$ is the number of ways to distribute the "exceptional" s_1-m_1 weight-1-elements between m blocks;
- $C=\frac{(s_2+s_3+\dots+s_p)!}{s_2!s_3!\dots s_p!}d_2^{s_2}\cdots d_p^{s_p}$ is the number of ways to form a chain out of s_2 elements of weight 2 having d_2 possible colors, etc...
- $D=C_{s_2+s_3+\dots+s_p-1}^{m-m_1-1}=\frac{(s_2+s_3+\dots+s_p-1)!}{(m-m_1-1)!(s_2+s_3+\dots+s_p-m+m_1)!}$ is the number of ways to cut the chain introduced above into $m-m_1$ nonempty pieces.

A formula for the number of colored weighted heaps of total weight N

For a fixed $N, m, m_1, s_1, s_2, \dots s_p$, we have

$$3^m \frac{m!}{m_1!(m-m_1)!} \frac{(s_1-m_1+m-1)!}{(m-1)!(s_1-m_1)!} \frac{(s_2+s_3+...+s_p)!}{s_2!s_3!...s_p!} d_2^{s_2} \cdots d_p^{s_p} \times$$

$$\times \frac{(s_2+s_3+...+s_p-1)!}{(m-m_1-1)!(s_2+s_3+...+s_p-m+m_1)!} \approx$$

$$3^m \frac{(s_1-m_1+m)!}{m_1!(s_1-m_1)!(m-m_1)!} \frac{(s_2+s_3+...+s_p)!}{s_2!s_3!...s_p!} d_2^{s_2} \cdots d_p^{s_p} \times$$

$$\times \frac{(s_2+s_3+...+s_p)!}{(m-m_1)!(s_2+s_3+...+s_p-m+m_1)!} \approx$$

The numbers of vertically indecomposable flype-equivalence classes of types || and × (computation by Artem Aleshin):

$$\begin{aligned} d_1 &= 1 \\ d_2 &= 0 \\ d_3 &= 2 \\ d_4 &= 2 \\ d_5 &= 10 \\ d_6 &= 28 \\ d_7 &= 82 \\ d_8 &= 295 \\ d_9 &= 942 \\ d_{10} &= 3387 \\ d_{11} &> d_9 \\ d_{12} &> d_{10} \end{aligned}$$

Thank you