On λ-homomorphic skew braces

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1. V.G. Bardakov, M.V. Neshchadim, M.K. Yadav, *Computing skew left braces of small orders*. Internat. J. Algebra Appl., https://doi.org/10.1142/S0218196720500216.

2. V.G. Bardakov, M.V. Neshchadim, M.K. Yadav, *On λ-homomorphic skew braces*. arXiv:2004.05555v1 [math.RA] 12 Apr 2020

For a skew left brace $(G, \cdot, \circ)$, the map

$$\lambda : (G, \circ) \to \mathrm{Aut}(G, \cdot), \quad a \mapsto \lambda_a,$$

where $\lambda_a(b) = a^{-1} \cdot (a \circ b)$ for all $a, b \in G$, is a group homomorphism. Then λ can also be viewed as a map from $(G, \cdot)$ to $\mathrm{Aut}(G, \cdot)$, which, in general, may not be a homomorphism. We study skew left braces $(G, \cdot, \circ)$ for which $\lambda : (G, \cdot) \to \mathrm{Aut}(G, \cdot)$ is a homomorphism. Such skew left braces will be called λ-homomorphic [2].

Drienfild V.G.(1990) On some unsolved problems in quantum group theory. In Quantum groups (Leningrad, 1990), volume 1510 of Lecture Notes in Math., pages 1–8. Springer, Berlin, 1992.

formulated the next problem.

Problem. "The quantum Yang-Baxter equation

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$$

makes sense even if R is not linear operator $V \otimes V \longrightarrow V \otimes V$ but a map $X \times X \longrightarrow X \times X$ where X is a set. Of course to each set-theoretical solution to (1) there corresponds a solution in the usual sense (apply the functor $X \longrightarrow$ the free module generated by X). **Maybe it would be interesting to study set-theoretical solution to (1)**.

Example. (B.B.Venkov) If

$$R(x, y) = (x, x \circ y)$$

for some operation "$\circ$" on X then (1) is equivalent to the following distributivity identity:

$$x \circ (y \circ z) = (x \circ y) \circ (x \circ z).$$

Note that the set X with identity (2) is a **rack** (or in particular case is a **quandle**).

After this paper the problem of studying set-theoretical solution to (1) was intensively investigated by many authors.

D.Bachiller, A.Smoktunowicz, L.Guarnieri, L.Vendramin, V.Lebed, N.P.Byott F.Cedo, E.Jespers, J.Okninski, T.Gateva-Ivanova, L.N.Childs, P.Dehornoy, P.Etingof, S.Gelaki, R.Guralnick, J.Saxl, T.Schedler, A.Soloviev, R.Fenn, P.Cameron, S.Majid, C.Greither, B.Pareigis, P.Hubbert, T.Kohl, J.-H.Lu, M.Yan, Y.-C.Zhu, S.Nelson, W.Rump, M.Takeuchi and etc.

Yang-Baxter equation can be rewritten in equivalent form:

$$(r \times id)(id \times r)(r \times id) = (id \times r)(r \times id)(id \times r),$$

where $r : X \times X \longrightarrow X \times X$ is a bijective map. (Enough to put $r = \xi R$, где $\xi(x, y) = (y, x)$.)

W.Rump (2007) in the paper

W.Rump, Braces, radical rings, and the quantum Yang-Baxter equation. J.Algebra, 307(1):153–170, 2007.

introduced new algebraic systems

Definition. A **brace** (classical brace) is an algebraic system $\langle A, +, \circ \rangle$ with two operations "$+$", "$\circ$" such that

1) $\langle A, + \rangle$ is an abelian group and $\langle A, \circ \rangle$ is a some group,

2) for every elements $a, b, c \in A$ holds the next

$$a \circ (b + c) = (a \circ b) - a + (a \circ c).$$

W.Rump also observed that a classical brace produces an involutive non-degenerate solution of the Yang-Baxter equation

$$r_A : A \times A \longrightarrow A \times A,$$

where

$$r_A(a,b) = (ab - a, (ab - a)^{-1}ab).$$

Skew braces are generalization of classical braces. They were introduced in paper

L.Guarnieri, L.Vendramin, Skew braces and the Yang-Baxter equation. Math. Comp., 86(307):2519–2534, 2017.

for study of non-involutive solutions of the Yang-Baxter equation.

Definition. A (left) skew brace is a triple $\langle A, \cdot, \circ \rangle$ where $\langle A, \cdot \rangle$ and $\langle A, \circ \rangle$ are groups and the compatibility condition

$$a \circ (bc) = (a \circ b) a^{-1} (a \circ c)$$

holds for all $a, b, c \in A$, where a^{-1} denotes the inverse of a with respect to the group $\langle A, \cdot \rangle$. The group $\langle A, \cdot \rangle$ will be the **additive group** of the brace and $\langle A, \circ \rangle$ will be **multiplicative group** of the brace. A skew brace is said to be **classical** if its additive group is abelian.

Main property. Let A be skew brace. The map

$$\lambda : \langle A, \circ \rangle \longrightarrow \mathsf{Aut}\ \langle A, \cdot \rangle\,, \quad \lambda_a(b) = a^{-1}(a \circ b)$$

is a group homomorphism.

Theorem. Let A be a skew left brace. Then

$$r_A : A \times A \longrightarrow A \times A,$$

$$r_A(a,b) = (\lambda_a(b), \mu_b(a)) = (\lambda_a(b), \lambda^{-1}_{\lambda_a(b)}((a \circ b)^{-1} a (a \circ b))),$$

is a non-degenerate solution of the Yang-Baxter equation. Furthermore, r_A is involutive if and only if $ab = ba$ for all $a, b \in A$.

Definition. A **biquandle** is a non-degenerate set-theoretical solution (X, r) of the Yang-Baxter equation such that there exists a bijection $t : X \longrightarrow X$ such that $r(t(x), x) = (t(x), x)$ for all $x \in X$.

Proposition. Let A be skew brace and r_A its associated solution of the Yang-Baxter equation. Then (A, r_A) is a biquandle.

L.Guarnieri, L.Vendramin, Skew-braces and Yang-Baxter equation. Math.Comp., 86(2017), 2519–2534.

obtained characterization of skew left braces in terms of regular subgroups of holomorph.

Let A be a group. The **holomorph** of A is the group

$$\mathrm{Hol}\, A = \mathrm{Aut}\, A \ltimes A,$$

where the product is given by

$$(f, a)(g, b) = (fg, af(b))$$

for all $a, b \in A$ and $f, g \in \mathrm{Aut}\, A$. Any subgroup H of $\mathrm{Hol}\, A$ acts on A by the ruler

$$(f, a) \cdot b = af(b), \quad a, b \in A, \;\; f \in \mathrm{Aut}\, A.$$

A subgroup H of $\mathrm{Hol}\, A$ is **regular** if for each $a \in A$ there exists a unique $(f, x) \in H$ such that $xf(a) = 1$. It is mean that action of H on A is **exact** and **transitively**.

Lemma. Let A be a group and H is a regular subgroup of $\mathrm{Hol}\,A$. Then $\pi_2|_H : H \longrightarrow A$, $(f, a) \longmapsto a$ is bijective.

Theorem. Let A be skew left brace. Then $\{\,(\lambda_a, a) \mid a \in A\,\}$ is a regular subgroup of $\mathrm{Hol}\,A$. Conversely, if $\langle A, \cdot\rangle$ is a group and H is a regular subgroup of $\mathrm{Hol}\,A$ then A is a skew left brace with $\langle A, \circ\rangle \cong H$, where

$$a \circ b = af(b)$$

and $(\pi_2|_H)^{-1}(a) = (f, a) \in H$.

Proposition. Let A be a group. There exists a bijective correspondence between skew left brace structures over A and regular subgroup of $\mathrm{Hol}\,A$. Moreover, isomorphic skew braces structures over A correspond to conjugate subgroups of $\mathrm{Hol}\,A$ by elements of $\mathrm{Aut}\,A$.

Definition.

A skew brace $(G, \cdot, \circ)$ is said to be a λ-*homomorphic skew brace* if the map $\lambda : (G, \cdot) \to \mathrm{Aut}\,(G, \cdot)$, defined by $\lambda_a(b) = a^{-1}(a \circ b)$, $a, b \in (G, \cdot)$, is a homomorphism.

Theorem (Bardakov-Neshchadim-Yadav).
Let G be a group, $\lambda : G \to \mathrm{Aut}\,G$ be a homomorphism of G into the group of its automorphisms. The set

$$H_\lambda := \{ (\lambda_a, a) \,|\, a \in G \}$$

is a subgroup of $\mathrm{Hol}\,G = \mathrm{Aut}\,G \ltimes G$ if and only if

$$[G, \lambda(G)] := \{b^{-1}\lambda_a(b) \mid a, b \in G\} \subseteq \mathrm{Ker}\,\lambda.$$

Moreover, if H_λ is a subgroup, then it is regular, and therefore we get a skew brace $(G, \cdot, \circ)$, where '$\circ$' is defined by $a \circ b = a\lambda_a(b)$.

Definition. A skew-brace G is said to be *meta-trivial* if there exists a trivial sub-skew brace H of G such that the skew brace G/H is trivial too.

Theorem (BNY). Any λ-homomorphic skew-brace is meta-trivial.

It is natural to ask

Question. Is it true that every meta-trivial skew brace is λ-homomorphic?

In the following result we present a reduction argument, which allows us to verify conditions on generators of a given group and on the images of the generators under λ.

Theorem (BNY). Let G be a group generated by $X = \{x_i \mid i \in I\}$ and $\lambda : G \to \mathrm{Aut}\, G$, $a \mapsto \lambda_a$, a homomorphism such that $\lambda_{x_i} = \varphi_i$ for $i \in I$. If

$$x_i^{-1}\varphi_j(x_i) \in \mathrm{Ker}\lambda, \quad i, j \in I,$$

then

$$b^{-1}\lambda_a(b) \in \mathrm{Ker}\lambda, \quad a, b \in G.$$

Definition. A skew brace $(G, \cdot, \circ)$ is said to *symmetric* if $(G, \circ, \cdot)$ is also a skew brace.

Proposition. A skew brace $(G, \cdot, \circ)$ is symmetric if and only if

$$\bar{b} \circ (a \cdot b) \circ \overline{a} \in \mathsf{Ker}\,\lambda$$

for all $a, b \in G$, where $\overline{a}$ denotes the inverse of a under '$\circ$'.

Definition. A λ-homomorphic skew brace $(G, \cdot, \circ)$ is said to be a λ-*cyclic skew brace* if the image $\mathsf{Im}\,\lambda$ is a cyclic subgroup of $\mathsf{Aut}\,(G, \cdot)$.

Theorem (BNY). Every λ-cyclic skew brace is symmetric.

Examples of skew braces on free groups.

Let F_n be a free group with free generators $x_1, \ldots, x_n$ and

$$\mathrm{Hol}\, F_n = \mathrm{Aut} F_n \ltimes F_n = \{\, (f, a) \mid f \in \mathrm{Aut} F_n,\ a \in F_n \,\}$$

is its holomorph with operation

$$(f, a)(g, b) = (fg, af(b)).$$

Fix some homomorphism

$$\theta : F_n \longrightarrow \mathrm{Aut}\, F_n$$

and consider the subset

$$H_\theta = \{\, (\theta(a), a) \mid a \in F_n \,\}$$

of $\mathrm{Hol}\, F_n$.

The set H_θ is a subgroup of $\mathrm{Hol}\, F_n$ if and only if

$$\theta(b^{-1}\theta(a)(b)) = 1$$

for all $a, b \in F_n$.

Example. Let F_2 be a free group with free generators x, y, θ is automorphism of F_2 such that $\theta : x \leftrightarrows y$ and $H_0 \leq F_2$ is the kernel of homomorphism $F_2 \longrightarrow \mathbb{Z}_2 = \langle z \rangle$, $x, y \longmapsto z$, $z^2 = 1$.

Then on the set F_2 we can define operation $\circ$ by the ruler

$$a \circ b = \begin{cases} ab, & \text{if } a \in H_0,\ b \in F_2, \\ a\theta(b), & \text{if } a \notin H_0,\ b \in F_2 \end{cases}$$

such that the algebraic system $\langle F_2, \cdot, \circ \rangle$ is a skew left brace, that is $\langle F_2, \circ \rangle$ is a group and for all $a, b, c \in F_2$

$$a \circ (b \cdot c) = (a \circ b) \cdot a^{-1} \cdot (a \circ c).$$

The group $\langle F_2, \circ \rangle$ is a free group with free generators x, y.

Example. Let F_2 be a free group with free generators x, y, θ is automorphism of F_2 such that $\theta : x \longmapsto x^{-1}$, $y \longmapsto y^{-1}$ and $H_0 \leq F_2$ is the kernel of homomorphism $F_2 \longrightarrow \mathbb{Z}_2 = \langle z \rangle$, $x, y \longmapsto z$, $z^2 = 1$.

Then on the set F_2 we can define operation $\circ$ by the ruler

$$a \circ b = \begin{cases} ab, & \text{if } a \in H_0,\ b \in F_2, \\ \\ a\theta(b), & \text{if } a \notin H_0,\ b \in F_2 \end{cases}$$

such that the algebraic system $\langle F_2, \cdot, \circ \rangle$ is a skew left brace, that is $\langle F_2, \circ \rangle$ is a group and for all $a, b, c \in F_2$

$$a \circ (b \cdot c) = (a \circ b) \cdot a^{-1} \cdot (a \circ c).$$

The group $\langle F_2, \circ \rangle$ is isomorphic to group

$$\Big\langle\, p, q, r, s \,\|\, s^2 = 1,\ sps = p^{-1},\ sqs = q^{-1},\ srs = pr^{-1}p^{-1} \,\Big\rangle \cong F_3 \leftthreetimes \mathbb{Z}_2.$$

V.G. Bardakov, M.V. Neshchadim, M. Singh, *Exterior and symmetric (co)homology of groups*. International Journal of Algebra and Computation. 2020

We introduce symmetric homology of groups and compute exterior and symmetric (co)homologies of some finite groups. We also compare the classical, exterior and symmetric (co)homologies. Finally, we derive restriction and corestriction homomorphisms for exterior cohomology.

Thank you!