

Euler characteristics of positively graded Lie algebras and combinatorial identities

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Groups and quandles in low dimensional topology

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Let us consider the cochain complex of a Lie algebra \mathfrak{g}

$$\mathbb{K} \xrightarrow{d_0=0} \mathfrak{g}^* = C^1(\mathfrak{g}) \xrightarrow{d_1} C^2(\mathfrak{g}) \xrightarrow{d_2} \dots \xrightarrow{d_{p-1}} C^p(\mathfrak{g}) \xrightarrow{d_p} \dots$$

$C^p(\mathfrak{g})$ is the vector space of continuous skew-symmetric p -linear forms with the differential d_p

$$d_p c(X_1, \dots, X_{p+1}) = \sum_{1 \leq i < j \leq p+1} (-1)^{i+j-1} c([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots)$$

The differential $d_1 : \mathfrak{g}^* \rightarrow \Lambda^2(\mathfrak{g}^*)$ coincides with the dual mapping of the Lie bracket $[\cdot, \cdot] : \Lambda^2 \mathfrak{g} \rightarrow \mathfrak{g}$ and

$$d(\rho \wedge \eta) = d\rho \wedge \eta + (-1)^{\deg \rho} \rho \wedge d\eta, \quad \forall \rho, \eta \in \Lambda^*(\mathfrak{g}^*).$$

Definition

The cohomology of $(C^*(\mathfrak{g}), d)$ is called the cohomology (with trivial coefficients) of the Lie algebra \mathfrak{g} and is denoted by $H^*(\mathfrak{g})$.

Dixmier's theorem

Theorem

Dixmier Let \mathfrak{g} be a n -dimensional nilpotent Lie algebra. Then

$$b_i(\mathfrak{g}) = \dim H^i(\mathfrak{g}) \geq 2, i = 1, \dots, n-1, \\ \dim H^0(\mathfrak{g}) = \dim H^n(\mathfrak{g}) = 1.$$

The last property is important, an arbitrary n -dimensional nilpotent Lie algebra admits a volume form $\Omega = e^1 \wedge \dots \wedge e^n, [\Omega] \neq 0$.

We have Poincaré duality in the cohomology and hence

$$\dim H^i(\mathfrak{g}) = \dim H^{n-i}(\mathfrak{g}).$$

Also we have $\dim H^*(\mathfrak{g}) \geq 2 \dim \mathfrak{g}$

Dixmier estimates are sharp, there are nilpotent Lie algebras with $\dim H^i(\mathfrak{g}) = 2$, $0 < i < \dim \mathfrak{g}$ in dimensions 3, 4, 6.

In dimension 5 however $\dim H^*(\mathfrak{g}) \geq 12$.

It seems that in higher dimensions > 6 Dixmier's estimate is not best possible (Conjecture).

Conjecture (Toral rank conjecture or Puppe conjecture)

Let \mathfrak{g} be a finite-dimensional nilpotent Lie algebra, then

$$\dim H^*(\mathfrak{g}) \geq 2^{\dim \mathfrak{z}},$$

where $\mathfrak{z} \subset \mathfrak{g}$ is the center of \mathfrak{g} .

The positive part W^+ of the Witt algebra

Definition

A Lie algebra \mathfrak{g} is called \mathbb{N} -graded, if

$$\mathfrak{g} = \bigoplus_i \mathfrak{g}_i, \quad i \in \mathbb{N}, \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, \quad \forall i, j \in \mathbb{N}.$$

The positive part W^+ of the Witt algebra can be defined by its basis e_1, e_2, e_3, \dots and the commutation relations

$$[e_i, e_j] = (j - i)e_{i+j}, \quad \forall i, j \in \mathbb{N}.$$

$$W^+ = \langle e_i = x^{i+1} \frac{d}{dx}, \quad i \in \mathbb{N} \rangle.$$

Consider $\mathfrak{g} = \bigoplus_{i=1}^{\infty} \mathfrak{g}_i$ of width one, i.e. $\dim \mathfrak{g}_i = 1, \forall i$.

Consider a second grading of the cochain complex

$$\Lambda^*(\mathfrak{g}^*) = \hat{\bigoplus}_k \Lambda_k^*(\mathfrak{g}),$$

where $\Lambda_k^q(\mathfrak{g})$ is spanned by q -forms

$$e^{i_1} \wedge \dots \wedge e^{i_q}, i_1 < \dots < i_q, i_1 + \dots + i_q = k.$$

The second grading is compatible with the differential d and with the exterior product and it induces a bigraded product structure in the cohomology $H^*(\mathfrak{g})$:

$$H_k^q(\mathfrak{g}) \wedge H_l^p(\mathfrak{g}) \rightarrow H_{k+l}^{q+p}(\mathfrak{g}).$$

The cochain complex $\Lambda^*(W^+)$

$$\begin{aligned}de^1 &= 0, de^2 = 0, \\de^3 &= e^1 \wedge e^2, de^4 = 3e^1 \wedge e^3, \\de^5 &= 3e^1 \wedge e^4 - e^2 \wedge e^3, \\&\dots\end{aligned}$$

One has to use the bigraded structure of d

$$\begin{aligned}0 &\rightarrow e^1 \rightarrow 0, \\0 &\rightarrow e^2 \rightarrow 0, \\0 &\rightarrow e^3 \rightarrow e^1 \wedge e^2 \rightarrow 0, \\0 &\rightarrow e^4 \rightarrow 2e^1 \wedge e^3 \rightarrow 0 \\0 &\rightarrow e^5 \rightarrow 3e^1 \wedge e^4, e^2 \wedge e^3 \rightarrow 0, \\0 &\rightarrow e^6 \rightarrow 4e^1 \wedge e^5 + 2e^2 \wedge e^4, e^2 \wedge e^4 \rightarrow -2e^1 \wedge e^2 \wedge e^3 \rightarrow 0, \\&\dots\end{aligned}$$

We have homogeneous cocycles $e^1, e^2, e^2 \wedge e^3, \dots$

Theorem (Goncharova, 1973)

$b^q(W^+) = \dim H^q(W^+) = 2$, for every $q \geq 1$, more precisely

$$b_k^q(W^+) = \dim H_k^q(W^+) = \begin{cases} 1, & \text{if } k = \frac{3q^2 \pm q}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

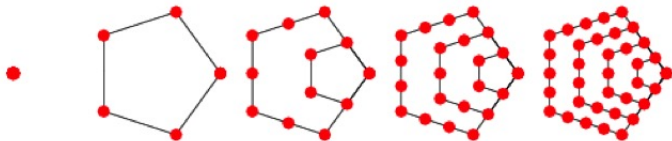
The numbers $\frac{3q^2 \pm q}{2}$ are so called Euler pentagonal numbers.

$$H^1(W^+) = \langle e^1, e^2 \rangle, \quad q = 1, \frac{3 \cdot 1^2 \pm 1}{2} = \mathbf{1, 2};$$

$$H^2(W^+) = \langle e^2 \wedge e^3, 3e^3 \wedge e^4 - e^2 \wedge e^5 \rangle, \quad q = 2, \frac{3 \cdot 2^2 \pm 2}{2} = \mathbf{5, 7};$$

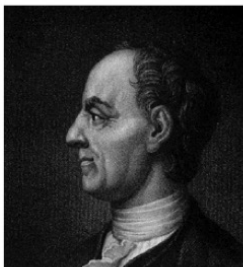
Pentagonal numbers

- Generalized pentagonal number $\frac{1}{2}(3m^2 \pm m)$
- First values
- **1**, 2, **5**, 7, **12**, 15, **22**,
26, **35**, ...



The Euler identity

$$\prod_{k=1}^{+\infty} (1 - x^k) = 1 + \sum_{m=1}^{+\infty} (-1)^m \left(x^{\frac{1}{2}(3m^2-m)} + x^{\frac{1}{2}(3m^2+m)} \right)$$



Right hand side of Euler identity (Betti numbers)

Consider the Euler characteristic of each subcomplex $\Lambda_k^*(W^+)$

$$\chi_k(W^+) = \sum_q (-1)^q b_k^q(W^+)$$

and take the corresponding generating function $\chi_{W^+}(t)$

$$\sum_{k=0} \chi_k(W^+) t^k = \sum_{k,q \geq 0} (-1)^q b_k^q(W^+) t^k = \sum_{k=0}^{\infty} (-1)^k \left(t^{\frac{3k^2-k}{2}} + t^{\frac{3k^2+k}{2}} \right).$$

Left hand side of Euler identity (cochains and partitions)

$e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_q}$ from $\Lambda_k^q(W^+)$ corresponds to a partition

$$i_1 + i_2 + \cdots + i_q = k, \quad 1 \leq i_1 < i_2 < \cdots < i_q.$$

and $\dim \Lambda_k^q(W^+) = V_q(k)$, where $V_q(k)$ is the number of partitions of a positive integer k into q **distinct** parts.

$$\chi_k(W^+) = \sum_{q \geq 0} (-1)^q \dim \Lambda_k^q(W^+) = V_{\text{even}}(k) - V_{\text{odd}}(k)$$

where $V_{\text{even}}(k)$ ($V_{\text{odd}}(k)$) is the number of partitions of $k > 0$ into even (odd) number of distinct parts.

$$\chi_{W^+}(t) = \sum_{k=0} \chi_k(W^+) t^k = \sum_k (V_{\text{even}}(k) - V_{\text{odd}}(k)) t^k = \prod_{i=1}^{+\infty} (1 - t^i).$$

Consider a finite-dimensional quotient

$W_n^+ = W^+ / \langle e_{n+1}, e_{n+2}, \dots \rangle$. It is \mathbb{N} -graded $W_n^+ = \bigoplus_{i=1}^n \langle e_i \rangle$.

We have the same cochain complex with differential d but with finite number e^1, e^2, \dots, e^n

$$de^k = \sum_{i < j, i+j=k} (j-i)e^i \wedge e^j$$

Victor Buchstaber, “Polynomial Euler characteristic of nilmanifolds”, *Funct. Anal. Appl.*, **58**:1 (2024), 17–33

Difference. Negative weights (second grading) $w(e_i) = -2i$.

The generating function $\chi_{W_n^+}(t)$ is now a polynomial
(polynomial Euler characteristic according to Buchstaber)

$$\chi_{W_n^+}(t) = \prod_{i=1}^n (1 - t^i) = \dots?$$

Ch. Deninger, W. Singhof, "On the cohomology of nilpotent Lie algebras", Bull. Soc. math. France, **116**, 1988, 3–14.

They consider the polynomial characteristic $P_{\mathfrak{g}}(t)$ of a finite-dimensional \mathbb{N} -graded Lie algebra $\mathfrak{g} = \bigoplus_{j=1}^n \mathfrak{g}_j$

$$P_{\mathfrak{g}}(t) = \prod_{i=1}^{+\infty} (1 - t^i)^{\dim \mathfrak{g}_i}$$

They apply an estimate

$$\dim H^*(\mathfrak{g}) \geq \text{the length of } P_{\mathfrak{g}}(t).$$

where the length of the polynomial $P_{\mathfrak{g}}(t)$ is defined as the sum of the absolute values of its coefficients.

They proved that 1) for graded Lie algebras $\mathfrak{g} = \bigoplus_{j=1}^n \mathfrak{g}_j$ with n fixed, total cohomology $\dim H^*(\mathfrak{g})$ grows exponentially with $\dim \mathfrak{g}$ and 2) TRC-conjecture for 2-step nilpotent Lie algebras.

Consider again

$$\chi_{W_n^+}(t) = \prod_{i=1}^n (1 - t^i) = \sum_{k,q=0} (-1)^q b_k^q(W_n^+) t^k$$

What we know about the cohomology $H^*(W_n^+)$?

"Что мы знаем о Басе? Ничего! И то не всё"

$$H^1(W_n^+) = \langle e^1, e^2 \rangle.$$

Moreover all other cocycles are of weights ≥ 2 . Hence it reflects in $\prod_{i=1}^n (1 - t^i)$ in following way

$$\prod_{i=1}^n (1 - t^i) = 1 - t^1 - t^2 + \dots$$

$$H^2(W_n^+) = ?$$

Cocycles $g_5 = e^2 \wedge e^3$ and $g_7 = e^2 \wedge e^5 - 3e^3 \wedge e^4$ exist if $n \geq 5$

$$\prod_{i=1}^5 (1 - t^i) = 1 - t - t^2 + t^5 + t^6 + t^7 - t^8 - t^9 - t^{10} + t^{13} + t^{14} - t^{15}$$

$-t^{15}$ corresponds to the volume form $\Omega_5 = e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^5$ of degree 5 and weight $1 + 2 + 3 + 4 + 5 = 15$.

What about the weight 6? Recall subcomplex $\Lambda_6^*(W^+)$

$$0 \rightarrow e^6 \rightarrow 4e^1 \wedge e^5 + 2e^2 \wedge e^4, e^2 \wedge e^4 \rightarrow -2e^1 \wedge e^2 \wedge e^3 \rightarrow 0,$$

we have no 1-form e^6 in our complex $\Lambda^*(W_5^+)$, hence 2-form $\omega_6 = 4e^1 \wedge e^5 + 2e^2 \wedge e^4$ is non exact!

$$\begin{aligned} \chi_{W_{15}^+}(t) &= \prod_{i=1}^{15} (1 - t^i) = \\ & 1 - t - t^2 + t^5 + t^7 - t^{12} - t^{15} + t^{16} - t^{18} - t^{19} - t^{20} + t^{22} + t^{23} + \\ & + t^{24} + t^{25} + 2t^{26} + t^{27} - t^{31} - t^{33} - t^{34} - 3t^{35} - 2t^{36} - 2t^{37} - t^{38} - \\ & - t^{40} + t^{41} + 2t^{42} + 3t^{43} + 2t^{44} + 3t^{45} + 2t^{46} + 2t^{47} + 2t^{48} + t^{49} - \\ & - 3t^{52} - 3t^{53} - 3t^{54} - 3t^{55} - 4t^{56} - 2t^{57} - 2t^{58} - t^{59} + t^{61} + 2t^{62} + \\ & + 2t^{63} + 4t^{64} + 3t^{65} + 3t^{66} + 3t^{67} + 3t^{68} - t^{71} - 2t^{72} - 2t^{73} - 2t^{74} \\ & - 3t^{75} - 2t^{76} - 3t^{77} - 2t^{78} - t^{79} + t^{80} + t^{82} + 2t^{83} + 2t^{84} + 3t^{85} + \\ & + t^{86} + t^{87} + t^{89} - t^{93} - 2t^{94} - t^{95} - t^{96} - t^{97} - t^{98} + t^{100} + t^{101} + \\ & + t^{102} - t^{104} + t^{105} + t^{108} - t^{113} - t^{115} + t^{118} + t^{119} - t^{120} \end{aligned}$$

We see the cocycle $\omega_{16} = 14e^1 \wedge e^{15} + 12e^2 \wedge e^{14} + \dots + 2e^7 \wedge e^9$

What about the next three terms $-t^{18} - t^{19} - t^{20}$ from the polynomial $\chi_{W_{15}^+}(t)$?

Let $n \geq 15$. Then

- $H^1(W_n^+) = \langle e^1, e^2 \rangle$;
- $H^2(W_n^+) = \langle g_5, g_7, \omega_{n+1} \rangle$, where
 $\omega_{n+1} = (n-1)e^1 \wedge e^n + (n-3)e^2 \wedge e^{n-1} + \dots = de^{n+1}$
- $H^3(W_n^+) = \langle g_{12}, g_{15}, e^2 \wedge \omega_{n+1}, \rho_{n+4}, \rho_{n+5} \rangle$, where
 $\rho_{n+4} = e^2 \wedge \xi_{n+2} - ne^3 \wedge \omega_{n+1}$,
 $\rho_{n+5} = e^2 \wedge \xi_{n+3} - (n+1)e^3 \wedge \xi_{n+2} + \frac{n(n+1)}{2}e^4 \wedge \omega_{n+1}$,
 ξ_{n+2}, ξ_{n+3} are projections of de^{n+2}, de^{n+3} to $\Lambda^2(W_n^+)$

$$\xi_{n+2} = (n-2)e^2 \wedge e^n + (n-4)e^3 \wedge e^{n-1} + \dots;$$

$$\xi_{n+3} = (n-3)e^3 \wedge e^n + (n-5)e^4 \wedge e^{n-1} + \dots$$

Hence $b_1(W_n^+) = 2, b_2(W_n^+) = 3, b_3(W_n^+) = 5$.

```
> f(t) := product( 1-t^k, k=1..n );
```

$$f(t) := \prod_{k=1}^n (1-t^k)$$

```
> n:=1000:f(t):=product( 1-t^k, k=1..n );
```

```
> taylor(f(t), t=0, 120 );
```

$$1 - t - t^2 + t^5 + t^7 - t^{12} - t^{15} + t^{22} + t^{26} - t^{35} - t^{40} + t^{51} + t^{57} - t^{70} - t^{77} + t^{92} + t^{100} - t^{117} + O(t^{120})$$

```
> n:=105:f(t):=product( 1-t^k, k=1..n );
```

```
> taylor(f(t), t=0, 120 );
```

$$1 - t - t^2 + t^5 + t^7 - t^{12} - t^{15} + t^{22} + t^{26} - t^{35} - t^{40} + t^{51} + t^{57} - t^{70} - t^{77} + t^{92} + t^{100} + t^{106} - t^{108} - t^{109} - t^{110} + t^{113} + t^{114} + t^{115} + t^{116} + O(t^{120})$$

Le term t^{106} corresponds to Ω_{106}

Trois terms $-t^{108}-t^{109}-t^{110}$ correspond aux 3 forms dans H^3

Fibonacci conjecture

Conjecture (M., 2000)

Stable Betti number $b_q = \dim(H^q(W_n^+))$, $n > 3q - 1$ is equal to $(q + 2)$ -th Fibonacci number F_{q+2}

$$b_q = \dim(H^q(W_n^+)) = F_{q+2}, \quad n > 3q + 1.$$

Фестиваль Света в Турине





Dmitry Millionshchikov

Euler characteristics of positively graded Lie algebras and combinatorial identities

Consider a Lie subalgebra $W^+(k) = \langle e_k, e_{k+1}, e_{k+2}, \dots \rangle$.

Theorem (Gontcharova, 1973)

$$H^q(W^+(k)) = \binom{q+k-1}{k-1} + \binom{q+k-2}{k-1}.$$

Let consider the Serre-Hochschild spectral sequence of the pair $(W^+, W^+(n+1))$ that converges to the $H^*(W^+)$.
 The first row of E_2 is our answer. Consider $H^q(L_1/L_{n+1})$ only for $n \gg q$.

Proposition

$H^q(W^+) = E_\infty^{q,0} = E_3^{q,0} = H^q(W^+)$. In other words:
 Gontcharova's cocycles of small degrees q are in the first row of E_2 and survive till E_∞ .

$$E_3 = E_4 = \dots = E_\infty.$$

0	0	0	0	...
0	0	0	0	...
0	e^1, e^2	g_5, g_7	g_{12}, g_{15}	...

The term E_2 of the spectral sequence.

$e^{n+1} \wedge e^{n+4}$				
	d_2		d_2	
e^{n+1}	$e^2 \wedge e^{n+1}$ $e^2 \wedge e^{n+2} + \dots$ $e^2 \wedge e^{n+3} + \dots$	$g_5 \wedge e^{2n} + \dots$ $g_7 \wedge e^{n+1}$ $g_7 \wedge e^{n+5} + \dots$ $\omega_{n+1} \wedge e^{n+1}$		
0	e^1, e^2	ω_{n+1} g_5, g_7	$e^2 \wedge \omega_{n+1}$ $e^2 \wedge \xi_{n+2} + \dots$ $e^2 \wedge \xi_{n+3} + \dots$ g_{12}, g_{15}	$g_7 \wedge \omega_{n+1}$ \dots $g_7 \wedge \xi_{n+5} + \dots$ $\omega_{n+1}^2, g_{22}, g_{26}$

Статья в Кванте Д.Б.Фукса

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Д. Фукс

О раскрытии скобок, об Эйлере, Гауссе, Макдональде и об упущенных возможностях

Сколько раз ждешь от нас дождаться раскрытия скобок и приведения? Тысячи, и может быть, десятки тысяч! Если в это и мне кажется что-нибудь привлекательным, то это надежда, что результат удивителен, если прояснит некоторые черты, примет благоприятный вид, как, скажем,

$$\frac{(d+1)(d-1)-d^2-d^4}{(1-d)(1+d-d^2-d^4)} = 1-d^{2d}$$

Никто не ждет, что в подобных равенствах, только горами выходящих и гораций более глубоких. Они составляют результат более или менее сложной работы архаичной математической мысли. Слова математики и поговорки возвращаются звучной и благозвучной и повторяют за мной все выкладки: что означает не только запись, но и содержание статьи, но и значение степени истинности ее результатов.

1. Топология Эйлера

В середине XVIII века — это было в 1786 году или несколько раньше — Леонард Эйлер изобрел формулу коэффициента топологии

$$\chi_g(X) = (1-g)(1-d^2)(1-d^2-1-d^4)$$

Он раскрыл скобки и произвел — и получил поразительный ре-

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“О раскрытии скобок,
об Эйлере, Гауссе,
Макдональде и об
упущенных
возможностях”.

Журнал «Квант», 1981
год, номер 8.

Symplectic case $n = 2k$

The closed 2-form

$$\omega_{2k+1} = de^{2k+1} = (2k-1)e^1 \wedge e^{2k} + (2k-3)e^2 \wedge e^{2k-1} + \dots + e^k \wedge e^{k+1}$$

is symplectic, all its powers ω_{2k+1}^m are non-trivial cocycles.

The cohomology algebra $H^*(W_{2k}^+)$ does not satisfy **hard Lefschetz condition**

$$[e^1 \wedge \omega_{n+1}] = 0, \quad e^1 \wedge \omega_{n+1} = \frac{1}{n} d\xi_{n+2},$$

where

$$\xi_{n+2} = (n-2)e^2 \wedge e^n + (n-4)e^3 \wedge e^{n-1} + \dots = de^{n+2} - ne^1 \wedge e^{n+1}.$$

Hence $d\xi_{n+2} = ne^1 \wedge de^{n+1} = ne^1 \wedge \omega_{n+1}$.

Example

A Lie algebra $\mathfrak{m}_0(n)$ is defined by its basis e_1, e_2, \dots, e_n and commutation relations

$$[e_1, e_i] = \begin{cases} e_{i+1}, & i + 1 \leq n, \\ 0, & i + 1 > n. \end{cases}$$

It is also \mathbb{N} -graded Lie algebra with 1-dimensional homogeneous components $\langle e_i \rangle$ and hence

$$\chi_{\mathfrak{m}_0(n)}(t) = \prod_{i=1}^n (1 - t^i) = \sum_{k,q=0} (-1)^q b_k^q(\mathfrak{m}_0(n)) t^k$$

Consider $(n-1)$ -dimensional $\mathfrak{sl}(2, \mathbb{K})$ -module $V(\lambda)$, $\lambda = n-2$ by its basis $\{f_i, 0 \leq i \leq n-2\}$ and

$$\begin{aligned}Hf_i &= (\lambda - 2i)f_i, \\Yf_i &= (i+1)f_{i+1}, \\Xf_i &= (\lambda - i + 1)f_{i-1},\end{aligned}\tag{1}$$

where we set $f_{-1} = f_{n-1} = 0$. The $\mathfrak{sl}(2, \mathbb{K})$ -module $V(n-2)$ is generated by its primitive element f_0 of weight $\lambda = n-2$:

$$Hf_0 = \lambda f_0, Xf_0 = 0, f_i = Y^i f_0 / i!, i = 1, \dots, n-2.$$

We realize $V(n-2) = \langle e^2, \dots, e^n \rangle$ with $X = ad^* e_1, Xe^i = e^{i+1}$.

$$e^2 = f_0, e^{i+2} = f_i / \prod_{l=1}^i (n-1-l), i = 1, \dots, n-2$$

Bordemann reduced the computation of $\dim H^q(\mathfrak{m}_0(n))$ to the problem of finding $\text{Ker} X_q : \Lambda^q(V(\lambda)) \rightarrow \Lambda^q(V(\lambda))$

$$\dim H^q(\mathfrak{m}_0(n)) = \dim \text{Ker} X_q + \dim \text{Ker} X_{q-1}.$$

The dimension of $\text{Ker} X_q$ is equal to the **number of irreducible $\mathfrak{sl}(2, \mathbb{K})$ -modules in the decomposition of $\Lambda^q(V(n-2))$** and the last number is equal to the dimension of the zero-eigenspace plus the dimension of one-eigenspace of $H|_{\Lambda^q(V(n-2))}$

$$H(e^{2+i_1} \wedge \dots \wedge e^{2+i_q}) = (q(n-2) - 2 \sum_{l=1}^q i_l) e^{2+i_1} \wedge \dots \wedge e^{2+i_q}.$$

$\dim \text{Ker} X_q =$ number of solutions $(q(n-2) - 2 \sum_{l=1}^q i_l) = 0, 1$ or

$$\sum_{l=1}^q i_l = \left\lfloor \frac{q(n-2)}{2} \right\rfloor, \quad 0 \leq i_1 < \dots < i_q \leq n-2.$$

$$\dim \text{Ker}(D_1(n))_q = V_{q,n-1}\left(\left\lfloor \frac{qn}{2} \right\rfloor\right),$$

where $V_{q,n-1}(N)$ is the number of partitions of N into q distinct summands i_1, \dots, i_q such that $1 \leq i_1 < \dots < i_q \leq n-1$.

$$\dim H^q(\mathfrak{m}_0(n)) = V_{q,n-1}(\lfloor qn/2 \rfloor) + V_{q-1,n-1}(\lfloor (q-1)n/2 \rfloor).$$

It possible to write some values (Armstrong and Sigg, 1996)

$$\dim H^2(\mathfrak{m}_0(n)) = \left\lfloor \frac{n+1}{2} \right\rfloor, \quad \dim H^3(\mathfrak{m}_0(n)) = \left\lfloor \left(\frac{\binom{n+1}{2}}{2} + \frac{1}{8} \right) \right\rfloor,$$

$$\dim H^4(\mathfrak{m}_0(n)) = \left\lfloor \frac{4}{3} \left(\frac{\binom{n+1}{2}}{3} \right) + \frac{4n+13}{36} \right\rfloor.$$