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Triangular and near-trivial quandles

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Properties of triangular and near-trivial quandles are defined and studied.

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1. Quandles and Group Automorphisms

Definition 1. A quandle is an algebraic system $\langle Q; \circ, / \rangle$, with the multiplication “ \circ ” and division “ $/$ ” such that for all $x, y, z \in Q$ we have

(1) left solvability which is equivalent to division on the right, $(x \circ y) / y = x$, $(x / y) \circ y = x$,

(2) idempotency $x \circ x = x$,

(3) distributivity $(x \circ y) \circ z = (x \circ z) \circ (y \circ z)$.

If $\langle Q; \circ, / \rangle$ is a quandle then so is the algebraic system $\langle Q; /, \circ \rangle$ called the **inverse quandle**.

Remark. By definition, the mapping

$$I_y : x \mapsto x \circ y, \quad x, y \in Q$$

is an automorphism of the quandle Q (inner automorphism).

Definition 2. A quandle $\langle Q; \circ, / \rangle$ is **involutive** whenever the operations of multiplication and division coincide, i.e., given $x, y \in Q$, we have $x \circ y = x/y$, so that $(x \circ y) \circ y = x$.

Definition 3. A quandle $\langle Q; \circ, / \rangle$ is **abelian** whenever

$$(x \circ y) \circ (z \circ t) = (x \circ z) \circ (y \circ t)$$

for all $x, y, z, t \in Q$.

M. Hosszu (1957, 1959) showed that, given an arbitrary group $\langle G; \cdot, ^{-1}, e \rangle$ and using an arbitrary automorphism $\varphi \in \text{Aut}(G)$ we can construct the quandle with the operation

$$x \circ y = \varphi(xy^{-1})y, \quad x, y \in G.$$

If G is an abelian group then the resulting quandle is abelian. A quandle with such an operation is called a **generalized Alexander quandle**.

If we use the identity automorphism $\varphi = id$, then by using the resulting operation

$$x \circ y = (xy^{-1})y = x$$

we construct a **trivial quandle**. A trivial quandle can be constructed not only over a group, but also over an arbitrary set X .

2. Triangular Quandles

Recall that a set Q with a relation \leq defined on it is called **linearly ordered** if for any elements $x, y, z \in Q$

1) $x \leq y$ or $y \leq x$ holds,

2) if $x \leq y$ and $y \leq x$, then $x = y$,

3) if $x \leq y$ and $y \leq z$, then $x \leq z$.

Definition 4. A quandle $\langle Q; \circ; \leq \rangle$ over an ordered set Q will be called **lower triangular** (LT-quandle) if for arbitrary $x, y \in Q$ such that $x \leq y$, the equality $x \circ y = x$ holds.

Example multiplication table for a finite lower triangular quandle:

\circ	1	2	3	\dots	n
1	1	1	1	\dots	1
2	$*$	2	2	\dots	2
3	$*$	$*$	3	\dots	3
\vdots				\dots	\vdots
n	$*$	$*$	$*$	\dots	n

Upper-triangular quandles can be defined in a similar way.

Definition 5. A quandle $\langle Q; \circ; \leq \rangle$ over an ordered set Q will be called **upper-triangular** (UT-quandle), if for arbitrary $x, y \in Q$ such that $y \leq x$, the equality $x \circ y = x$ holds.

Example. On a set of one or two elements there exists only the trivial quandle. Joyce's quandle over a three-element set with a multiplication table

○		1	2	3
1		1	1	1
2		3	2	2
3		2	3	3

is the only non-trivial LT-quandle.

Remark. If on the set Q of the LT-quandle $\langle Q; \circ; \leq \rangle$ we define the inverse (dual) order

$$x \leq' y \Leftrightarrow y \leq x, \quad x, y \in Q,$$

then the quandle $\langle Q; \circ; \leq' \rangle$ will be upper triangular. Indeed,

$$x \circ y = x \Leftrightarrow x \leq y \Leftrightarrow y \leq' x.$$

Therefore, we will formulate further assertions for LT-quandles. Due to this remark, they are automatically valid for UT-quandles, taking into account the indicated duality.

Let x be some element of the lower triangular quandle $\langle Q; \circ; \leq \rangle$, denote

$$Q_{\geq x} = \{ y \in Q \mid y \geq x \}.$$

From the definition of an LT quandle it follows that $Q_{\geq x}$ is also an LT-quandle and, moreover, for any $x \leq y$ the inclusion $Q_{\geq x} \supseteq Q_{\geq y}$ holds. In a similar way, one can define a subset

$$Q_{> x} = \{ y \in Q \mid y > x \},$$

which is also an LT-quandle.

Remark. In the quandle $Q_{\geq x}$, the element x is a left zero, i.e. $x \circ y = x$ for any $y \in Q_{\geq x}$, and the sequence of nested subquandles $Q_{\geq x} \supseteq Q_{\geq y}$ corresponds to the sequence of left zeros $x \leq y$. (Note that the LT quandle $Q_{> x}$ may not have a left zero.) Unlike the linear order, the sequence of nested subquandles and the corresponding sequence of left zeros is **an invariant characteristic**.

We will use this observation to show that the direct product of two LT-quandles is not necessarily an LT-quandle.

Example. The direct product of two Joyce quandles.

Take two triangular quandles \mathcal{J}^1 and \mathcal{J}^2

\mathcal{J}^1	1	2	3
1	1	1	1
2	3	2	2
3	2	3	3

\mathcal{J}^2	a	b	c
a	a	a	a
b	c	b	b
c	b	c	c

and construct their direct product $\mathcal{J}^1 \times \mathcal{J}^2$:

$\mathcal{J}^1 \times \mathcal{J}^2$	$(1, a)$	$(1, b)$	$(1, c)$	$(2, a)$	$(2, b)$	$(2, c)$	$(3, a)$	$(3, b)$	$(3, c)$
$(1, a)$	$(1, a)$	$(1, a)$	$(1, a)$	$(1, a)$	$(1, a)$	$(1, a)$	$(1, a)$	$(1, a)$	$(1, a)$
$(1, b)$	$(1, c)$	$(1, b)$	$(1, b)$	$(1, c)$	$(1, b)$	$(1, b)$	$(1, c)$	$(1, b)$	$(1, b)$
$(1, c)$	$(1, b)$	$(1, c)$	$(1, c)$	$(1, b)$	$(1, c)$	$(1, c)$	$(1, b)$	$(1, c)$	$(1, c)$
$(2, a)$	$(3, a)$	$(3, a)$	$(3, a)$	$(2, a)$	$(2, a)$	$(2, a)$	$(2, a)$	$(2, a)$	$(2, a)$
$(2, b)$	$(3, c)$	$(3, b)$	$(3, b)$	$(2, c)$	$(2, b)$	$(2, b)$	$(2, c)$	$(2, b)$	$(2, b)$
$(2, c)$	$(3, b)$	$(3, c)$	$(3, c)$	$(2, b)$	$(2, c)$	$(2, c)$	$(2, b)$	$(2, c)$	$(2, c)$
$(3, a)$	$(2, a)$	$(2, a)$	$(2, a)$	$(3, a)$	$(3, a)$	$(3, a)$	$(3, a)$	$(3, a)$	$(3, a)$
$(3, b)$	$(2, c)$	$(2, b)$	$(2, b)$	$(3, c)$	$(3, b)$	$(3, b)$	$(3, c)$	$(3, b)$	$(3, b)$
$(3, c)$	$(2, b)$	$(2, c)$	$(2, c)$	$(3, b)$	$(3, c)$	$(3, c)$	$(3, b)$	$(3, c)$	$(3, c)$

The quandle $\mathcal{J}^1 \times \mathcal{J}^2$ has a single left zero $(1, a)$ and the subset $\mathcal{J}^1 \times \mathcal{J}^2 \setminus \{(1, a)\}$ no longer has left zeros. Therefore, by Remark 2, the quandle $\mathcal{J}^1 \times \mathcal{J}^2$ is not triangular.

Theorem 1. If $\langle Q, \circ \rangle$ is an LT-quandle, then the inner automorphisms $I_x, x \in Q$ satisfy the following relations

- 1) $I_y(x) = x$ for $y \geq x$;
- 2) $I_x I_y = I_y I_x$ for any $x, y \in Q$;
- 3) for arbitrary $x, y, z \in Q$, if $I_y(x) = z$, then $I_x = I_z$.

Conversely, if conditions 1), 2), 3) are satisfied for the permutations $I_x, x \in Q$ of the linearly ordered set $\langle Q, \leq \rangle$, then the operation $x \circ y = I_y(x), x, y \in Q$ defines the structure of an LT-quandle on the set Q .

Let us show that there always exists an LT-quandle with a single arbitrary column. Namely, the following lemma holds.

Lemma 1. Let $\langle Q; \leq \rangle$ be an arbitrary linearly ordered set, $a \in Q$ be some fixed element, and φ be an arbitrary bijection of $Q_{>a}$ onto itself. Then the operation

$$x \circ y = \begin{cases} x, & \text{if } y \neq a, \\ x, & \text{if } y = a, x \leq a, \\ \varphi(x), & \text{if } y = a, a < x \end{cases}$$

defines the structure of an LT-quandle on the set Q with a single non-trivial column in the multiplication table with index a .

Theorem 1 and Lemma 1 allow us to efficiently write lower triangular quandles over a finite number of symbols. As an example, we will describe LT-quandles on 4 elements. For substitutions, we will use the cyclic form of notation and, in fact, we begin the description by considering the substitution I_1 , that is, the first column of the multiplication table.

Example. LT-quandles on four elements.

Case 1. $I_1 = id$. There are two quandles

$$e \equiv \begin{array}{c|cccc} \circ & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 \end{array}, \quad b \equiv \begin{array}{c|cccc} \circ & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 4 & 3 & 3 \\ 4 & 4 & 3 & 4 & 4 \end{array},$$

which are actually obtained from LT-quandles on three elements by adding the left column and the top row.

Case 2. $I_1 = (234)$. By property 3) there is a relation $I_2 = I_3 = I_4$, therefore there is only one such quandle

$$a_1 \equiv \begin{array}{c|cccc} \circ & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 2 & 2 & 2 \\ 3 & 4 & 3 & 3 & 3 \\ 4 & 2 & 4 & 4 & 4 \end{array} .$$

Case 3. $I_1 = (243)$. By property 3) there is a relation $I_2 = I_3 = I_4$, therefore there is only one such quandle

$$a_5 \equiv \begin{array}{c|cccc} \circ & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 4 & 2 & 2 & 2 \\ 3 & 2 & 3 & 3 & 3 \\ 4 & 3 & 4 & 4 & 4 \end{array} .$$

Case 4. $I_1 = (23)$. By property 3) there is a relation $I_2 = I_3$, therefore there is only one such quandle

$$a_4 \equiv \begin{array}{c|cccc} \circ & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 2 & 2 & 2 \\ 3 & 2 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 \end{array} .$$

Case 5. $I_1 = (34)$. Due to property 3) there is a relation $I_2 = I_4$. For the substitution I_2 there are two possibilities $I_2 = id$ and $I_2 = (34)$, for which property 2) is satisfied. Therefore there are two quandles

$$a_2 \equiv \begin{array}{c|cccc} \circ & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 4 & 3 & 3 & 3 \\ 4 & 3 & 4 & 4 & 4 \end{array}, \quad b_2 \equiv \begin{array}{c|cccc} \circ & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 4 & 4 & 3 & 3 \\ 4 & 3 & 3 & 4 & 4 \end{array}.$$

Case 6. $I_1 = (24)$. For the substitution I_2 there are two possibilities $I_2 = id$ and $I_2 = (34)$. But $(24)(34) = (243) \neq (34)(24) = (234)$ therefore there is only one such quandle

$$a_3 \equiv \begin{array}{c|cccc} \circ & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 4 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 4 & 2 & 4 & 4 & 4 \end{array} .$$

3. Product of triangular quandles

For two right quasigroups on one set $Q^{(1)} = \langle Q; \circ_1 \rangle$ and $Q^{(2)} = \langle Q; \circ_2 \rangle$ we can define their product $Q^{(1)} \cdot Q^{(2)} = \langle Q; \circ_1 \circ_2 \rangle$, by the rule

$$x \circ_1 \circ_2 y = (x \circ_1 y) \circ_2 y.$$

Since this formula defines the composition of two permutations $I_y^{(1)}$ and $I_y^{(2)}$, the result will be a permutation on the set Q . Consequently, the operation defined in this way defines a right quasigroup on the set Q . The identity element will be the trivial quasigroup with the multiplication law

$$x \circ y = x, \quad x, y \in Q.$$

The set of all right quasigroups $P(Q)$, defined on a fixed set Q , forms a group that coincides with the Cartesian product of permutation groups of the set Q , indexed by the elements of the set Q :

$$P(Q) = \overline{\prod_{x \in Q} \text{Sym}(Q)},$$

where $\text{Sym}(Q)$ is the set of all bijections of the set Q onto itself.

If we consider a linearly ordered set $\langle Q; \leq \rangle$, then we can consider the LT-right quasigroups $P_T(Q)$. It is clear that with respect to the introduced operation they form a subgroup in the group $P(Q)$, which coincides with the Cartesian product

$$P_T(Q) = \overline{\prod_{x \in Q} \text{Sym}(Q_{>x})}.$$

where $\text{Sym}(Q_{>x})$ is the set of all bijections of the set $Q_{>x}$ onto itself.

By the above lemma, if we consider a subgroup generated by LT-quandles, then such a subgroup contains the direct product

$$\prod_{x \in Q} \text{Sym}(Q_{>x}).$$

In particular, for a finite set Q , $|Q| = n$ the group generated by LT-quandles coincides with this direct product and is isomorphic

$$S_{n-1} \times S_{n-2} \times \dots \times S_2 \times S_1.$$

Example. The group generated by quandles on four elements has the following multiplication table:

*	e	a_1	a_2	a_3	a_4	a_5	b	b_1	b_2	b_3	b_4	b_5
e	e	a_1	a_2	a_3	a_4	a_5	b	b_1	b_2	b_3	b_4	b_5
a_1	a_1	a_5	a_3	a_4	a_2	e	b_1	b_5	b_3	b_4	b_2	b
a_2	a_2	a_4	e	a_5	a_1	a_3	b_2	b_4	b	b_5	b_1	b_3
a_3	a_3	a_2	a_1	e	a_5	a_4	b_3	b_2	b_1	b	b_5	b_4
a_4	a_4	a_3	a_5	a_1	e	a_2	b_4	b_3	b_5	b_1	b	b_2
a_5	a_5	e	a_4	a_2	a_3	a_1	b_5	b	b_4	b_2	b_3	b_1
b	b	b_1	b_2	b_3	b_4	b_5	e	a_1	a_2	a_3	a_4	a_5
b_1	b_1	b_5	b_3	b_4	b_2	b	a_1	a_5	a_3	a_4	a_2	e
b_2	b_2	b_4	b	b_5	b_1	b_3	a_2	a_4	e	a_5	a_1	a_3
b_3	b_3	b_2	b_1	b	b_5	b_4	a_3	a_2	a_1	e	a_5	a_4
b_4	b_4	b_3	b_5	b_1	b	b_2	a_4	a_3	a_5	a_1	e	a_2
b_5	b_5	b	b_4	b_2	b_3	b_1	a_5	e	a_4	a_2	a_3	a_1

In this case, the right quasigroups

$$b_1 = \begin{array}{c|cccc} \circ & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 2 & 2 & 2 \\ 3 & 4 & 4 & 3 & 3 \\ 4 & 2 & 3 & 4 & 4 \end{array}, \quad b_3 = \begin{array}{c|cccc} \circ & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 4 & 2 & 2 & 2 \\ 3 & 3 & 4 & 3 & 3 \\ 4 & 2 & 3 & 4 & 4 \end{array},$$

$$b_4 = \begin{array}{c|cccc} \circ & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 2 & 2 & 2 \\ 3 & 2 & 4 & 3 & 3 \\ 4 & 4 & 3 & 4 & 4 \end{array}, \quad b_5 = \begin{array}{c|cccc} \circ & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 4 & 2 & 2 & 2 \\ 3 & 2 & 4 & 3 & 3 \\ 4 & 3 & 3 & 4 & 4 \end{array}.$$

are not quandles. Note that this group is isomorphic to the direct product $S_3 \times S_2$, where $S_3 = \{e, a_1, a_2, a_3, a_4, a_5\}$ and $S_2 = \{e, b\}$.

4. Extensions of triangular quandles

If there are two quandles Q , K such that $Q \cap K = \emptyset$, then we can pose the problem of defining the structure of a quandle on the set $Q \cup K$, with the following multiplication table

$$\begin{array}{c|c} Q & P \\ \hline R & K \end{array}$$

where the cell P consists of the elements of the quandle Q and corresponds to the multiplication of the elements of the quandle Q by the elements of the quandle K , and the cell R consists of the elements of the quandle K and corresponds to the multiplication of the elements of the quandle K by the elements of the quandle Q and the cells P , R are to be determined.

It is always possible to define cells P, R in a trivial way $k \circ q = k$ and $q \circ k = q$, $q \in Q$, $k \in K$. Naturally, the problem of classifying such quandle connections by cells P, R arises.

If there are two LT-quandles Q, K , $Q \cap K = \emptyset$, then on the set $Q \cup K$ one can introduce a natural order $q < k$, $q \in Q$, $k \in K$, preserving the existing orders of sets Q and K .

Then we can pose the problem of defining the structure of the LT-quandle on the set $Q \cup K$. Since the desired quandle is an LT-quandle, then the cell P corresponds to the trivial multiplication $q \circ k = q$, $q \in Q$, $k \in K$, and the cell R corresponds to the multiplication of the elements of the quandle K by the elements of the quandle Q and is subject to definition. As in the general case, it is possible to define the cell R by the trivial multiplication $k \circ q = k$, $q \in Q$, $k \in K$.

Example. Let's look at Joyce's two quandles

○		1	2	3		○		4	5	6
1		1	1	1		4		4	4	4
2		3	2	2	,	5		6	5	5
3		2	3	3		6		5	6	6

and we will make an LT-quandle out of them

o	1	2	3	4	5	6
1	1	1	1	1	1	1
2	3	2	2	2	2	2
3	2	3	3	3	3	3
4	*	*	*	4	4	4
5	*	*	*	6	5	5
6	*	*	*	5	6	6

If the fourth row contains elements other than 4, i.e. 5 or 6, then by property 3) of Theorem 1 it should be $I_4 = I_5$ or $I_4 = I_6$, and the corresponding columns should match. We have arrived at a contradiction. Consequently, only a table of the form is possible

○	1	2	3	4	5	6
1	1	1	1	1	1	1
2	3	2	2	2	2	2
3	2	3	3	3	3	3
4	4	4	4	4	4	4
5	*	*	*	6	5	5
6	*	*	*	5	6	6

To fill it, there are eight ways to place 5, 6 instead of stars. Therefore, from two Joyce quandles, it is possible to make 8 LT-quandles on six elements.

Let us note two special cases of the problem of expanding two LT-quandles, when one of the triangular quandles consists of a single element. In this case, the problem reduces to constructing a multiplication table with an additional first column and first row or last column and last row. This problem is solved efficiently using Theorem 1.

5. Near-trivial quandles

Recall that a quandle $\langle Q, \circ \rangle$ is called trivial if for arbitrary $x, y \in Q$ it holds

$$x \circ y = x.$$

By relaxing the requirement for a trivial quandle, we can consider quandles that are near-trivial.

Definition A quandle $\langle Q, \circ \rangle$ is called **near-trivial** if for arbitrary $x, y, z \in Q$ it holds

$$x \circ (y \circ z) = x \circ y.$$

Theorem 2. If $\langle Q, \circ \rangle$ is a near-trivial quandle, then the inner automorphisms I_x , $x \in Q$, corresponding to the elements of the quandle Q satisfy the relations

1) $I_x I_y = I_y I_x$ for any $x, y \in Q$;

2) for arbitrary $x, y, z \in Q$, if $I_y(x) = z$, then $I_x = I_z$.

Corollary. Triangular quandles are near-trivial.

Theorem (Birkhoff). For a non-empty class \mathcal{R} of algebraic systems to be a variety, it is necessary and sufficient that the following conditions be satisfied:

- a) the Cartesian product of an arbitrary sequence of \mathcal{R} -systems is an \mathcal{R} -system,
- b) any subsystem of an arbitrary \mathcal{R} -system is an \mathcal{R} -system,
- c) any homomorphic image of an arbitrary \mathcal{R} -system is an \mathcal{R} -system,

i.e. it is necessary and sufficient that the class of \mathcal{R} -systems be hereditary, multiplicatively and homomorphically closed.

An near-trivial quandle Q is defined by the identity

$$x \circ (y \circ z) = x \circ y, \quad x, y, z \in Q,$$

therefore the class of near-trivial quandles is closed under the operations of taking subquandles, homomorphic images and Cartesian products. Therefore, the following theorem holds.

Theorem 3. The class of near-trivial quandles is a manifold.

Remark. The example of the product of two Joyce quandles shows that the class of near-trivial quandles is strictly larger than the class of LT-quandles. Indeed, the class of near-trivial quandles is closed under direct products, but the class of LT-quandles is not.

Recall that a quandle $\langle Q; \circ, / \rangle$ is called abelian, if for arbitrary $x, y, z, t \in Q$ the identity

$$(x \circ y) \circ (z \circ t) = (x \circ z) \circ (y \circ t)$$

holds.

Proposition. Near-trivial quandles are abelian.

Recall that over an arbitrary group $\langle G; \cdot, ^{-1}, e \rangle$, using an arbitrary automorphism $\varphi \in \text{Aut}(G)$, one can construct a generalized Alexander quandle with the operation:

$$x \circ y = \varphi(xy^{-1})y, \quad x, y \in G.$$

Theorem 4. A generalized Alexander quandle constructed over a group $\langle G; \cdot, ^{-1}, e \rangle$ using an automorphism $\varphi \in \text{Aut}(G)$ defines a near-trivial quandle if and only if

$$\varphi(\varphi(t^{-1})t) = \varphi(t^{-1})t, \quad \forall t \in G.$$

Remark. 1) In the holomorph $\text{Hol } G$ of G , the relation $\varphi(\varphi(t^{-1})t) = \varphi(t^{-1})t$, $t \in G$ can be written in the commutator form $[\varphi, [\varphi, t]] = 1$.

2) Replacing t with t^{-1} , we rewrite the relation $\varphi(\varphi(t^{-1})t) = \varphi(t^{-1})t$, $t \in G$ as $\varphi(\varphi(t)t^{-1}) = \varphi(t)t^{-1}$. If we introduce the notation $\varphi(t) = at$, then we get that $\varphi(a) = a$. That is, the automorphism φ multiplies each element $t \in G$ from the left by an element from the subgroup

$$\text{Fix}(\varphi) = \{ a \in G \mid \varphi(a) = a \}.$$

Note that the element a , generally speaking, depends on the element t .

Let us give some examples.

Example. Let $G = \langle a, b \mid a^2 = b^2 = 1, ab = ba \rangle$ and $\varphi \in \text{Aut } G$ such that $\varphi(a) = b$, $\varphi(b) = a$. Then $\text{Fix}(\varphi) = \langle ab \rangle$ and the automorphism φ multiplies each element of G by a power of ab .

Example. Let G be a direct product of a free abelian group with free generators t_1, \dots, t_n , $n \geq 1$ and a periodic abelian group G_0 . Then any mapping φ of G defined on generators t_1, \dots, t_n by the equality $\varphi(t_k) = a_k t_k$, where a_k , $k = 1, \dots, n$ are some elements of G_0 and the identity on G_0 defines an automorphism of G . In this case $\text{Fix}(\varphi) \supseteq G_0$.

Example. As an example to Theorem 4, we can take an arbitrary nilpotent group of nilpotency class $k \geq 2$, $\gamma_k G \neq 1$, $\gamma_{k+1} G = 1$ and its arbitrary non-trivial inner automorphism induced by some element of $\gamma_{k-1} G$. In this case, the elements $\varphi(t)t^{-1}$, $t \in G$ lie in the center of G and the conjugation φ acts trivially on them.

Proposition. The generalized Alexander quandle constructed over the group $\langle G; \cdot, {}^{-1}, e \rangle$ by means of the automorphism $\varphi \in \text{Aut}(G)$ defines an LT-quandle if and only if φ is the identity automorphism.

Thank you!