On volumes of hyperbolic cone manifolds $5_2(\alpha)$ and $7_3^2(\alpha, \beta)$

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Groups and quandles in low-dimensional topology

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In begining of 1970's R. Riley in Southampton worked on representations of knots groups in $\mathrm{PSL}(2,\mathbb{C})$, which is the group of orientation preserving isometries of the hyperbolic space $\mathbb{H}^3$. After some time he got a faithful representation for the figure-eight knot 4_1, that the image was a discrete group and that the quotient of $\mathbb{H}^3$ by this group was the complement of knot 4_1. Thus, he discovered a **hyperbolic structure on the figure-eight knot complement**. This result by Riley was published much later, in 2013 (Riley died in 2000).

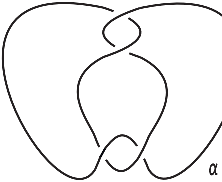

Fig.: Figure Eight knot 4_1

Then he showed that the same idea works for several other knots. In 1975 R. Riley found examples of **hyperbolic structures on some knot and link complements** in the three-dimensional sphere. Seven of them, so called **excellent knots**, were described later in his paper (1982).

Later, in the spring of 1977, W.P. Thurston announced an existence theorem for Riemannian metrics of constant negative curvature on 3-manifolds. In particular, it turned out that the **knot complement of a simple knot (excepting torical and satellite) admits a complete hyperbolic structure**[1]. This fact allowed to consider knot theory from the viewpoint of geometry and Kleinian group theory.

In 1980 W. Thurston constructed a **hyperbolic** 3-**manifold homeomorphic to the complement of knot** 4_1 in $\mathbb{S}^3$ by gluing faces of two regular ideal tetrahedra. This manifold has a complete hyperbolic structure.

In 1982 W. Thurston proposed his **Geometrization conjecture** about the existence of the geometrical structure on 3-manifolds. It implies several other conjectures, such as the Poincaré conjecture.

Both, the **Thurston geometrization conjecture and the Poincaré conjecture were proved by Grigori Perelman** in 2003.

[1] he wrote that he was motivated by Riley's beautiful examples.

The knot 5_2 is a rational knot of slope $7/2$

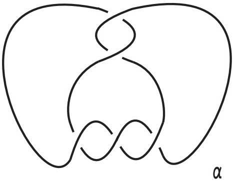

Fig.: Knot 5_2

Historically, this knot was the first one related with hyperbolic geometry. Indeed, it appeared as a singular set of the hyperbolic orbifold constructed by L.A. Best (1971) from a few copies of Lannér tetrahedra with Coxeter scheme $\circ \equiv \circ - \circ = \circ$. The fundamental set of this orbifold is a regular hyperbolic cube with dihedral angle $2\pi/5$.

In 2009, D. Gabai, R. Meyerhoff and P. Milley proved that the Weeks–Fomenko–Matveev manifold $\mathcal{M}_1$ of volume $0.9427\ldots$ is the smallest closed orientable hyperbolic 3-manifold.

It was proved by A. Vesnin and A. Mednykh (1998) that the manifold $\mathcal{M}_1$ is a cyclic 3-fold covering of the sphere $\mathbb{S}^3$ branched over the knot 5_2.

It was shown by J. Weeks's computer program SnapPea and proved by Moto-O Takahahsi (1985) that the complement $\mathbb{S}^3 \setminus 5_2$ is a union of three congruent ideal hyperbolic tetrahedra.

The next theorem has been proved by A. Rasskazov and A. Mednykh (1998), R. Shmatkov (2003) and J. Porti (2004) for the hyperbolic, Euclidian and spherical cases, respectively.

Theorem

The cone manifold $5_2(\alpha)$ *is hyperbolic for* $0 \leq \alpha < \alpha_0$, *Euclidean for* $\alpha = \alpha_0$, *and spherical for* $\alpha_0 < \alpha < 2\pi - \alpha_0$, *where* $\alpha_0 \simeq 2.40717$ *is a root of the equation*

$$-11 - 24\cos(\alpha) + 22\cos(2\alpha) - 12\cos(3\alpha) + 2\cos(4\alpha) = 0.$$

The following result is new.

Theorem (A. and Mednykh)

Let $5_2(\alpha),\ 0 \leq \alpha < \alpha_0$ *be a hyperbolic cone manifold. Then the volume of* $5_2(\alpha)$ *is given by the formula*

$$\mathrm{Vol}\, 5_2(\alpha) = i \int_{\bar{z}}^{z} \log \left(\frac{8(\zeta^2 + A^2)}{(1 + A^2)(1 - \zeta)(1 + \zeta)^2} \right) \frac{d\zeta}{\zeta^2 - 1},$$

where $A = \cot \frac{\alpha}{2}$ *and* z, $\Im(z) > 0$ *is a root of equation*

$$8(z^2 + A^2) = (1 + A^2)(1 - z)(1 + z)^2.$$

To prove this we use A-polynomial $A_{5_2}(L, M)$ for the knot 5_2. It has the form

$$L^3 M^{14} + L^2 \left(-M^{14} + 2M^{12} + 2M^{10} - M^6 + M^4\right) \\ + L \left(M^{10} - M^8 + 2M^4 + 2M^2 - 1\right) + 1.$$

By makin use the latter A-polynomial $A_{5_2}(L, M)$ we prove the following Cothangent rule for cone manifold $5_2(\alpha)$.

Theorem (A. and Mednykh)

Let γ_α be the complex length of the longitude for a hyperbolic cone manifold $5_2(\alpha)$ and $\gamma'_\alpha = \gamma_\alpha + 4i\,\alpha$. Then

$$\coth(\frac{\gamma'_\alpha}{4})\cot(\frac{\alpha}{2}) = i\,z,$$

where z, $\Im(z) < 0$, is a root of the equation $8(z^2 + A^2) = (1 + A^2)(1 - z)(1 + z)^2$ and $A = \cot(\frac{\alpha}{2})$.

Then using this Cothangent rule and the Schläfli formula we prove the above volume formula for a hyperbolic cone manifold $5_2(\alpha)$.

We illustrate the above result by the following two examples.

Example (1)

The volume of the hyperbolic orbifold $5_2(\frac{2\pi}{5})$ is given by the formula $\mathrm{Vol}\, 5_2(\frac{2\pi}{5}) =$

$$i \int\limits_{-2+\sqrt{5}-i\sqrt{-8+4\sqrt{5}}}^{-2+\sqrt{5}+i\sqrt{-8+4\sqrt{5}}} \log\left(\frac{8(\zeta^2+\cot(\frac{\pi}{5})^2)}{(1+\cot(\frac{\pi}{5})^2)(1-\zeta)(1+\zeta)^2}\right)\frac{d\zeta}{\zeta^2-1}.$$

By A. Best (1971), the fundamental polyhedron of the orbifold $5_2(\frac{2\pi}{5})$ is a regular $\frac{2\pi}{5}$-hexahedron formed by 48 hyperbolic Coxeter tetrahedra with the scheme $\circ \equiv \circ - \circ = \circ$, each having volume 0.0358850633. Hence, $\mathrm{Vol}\, 5_2(\frac{2\pi}{5}) = 48 \times 0.0358850633 = 1.7224830384$. The latter coincides with the above integral formula.

Example (2)

The volume of the hyperbolic orbifold $5_2(\frac{2\pi}{3})$ is given by the formula

$$\mathrm{Vol}\, 5_2(\frac{2\pi}{3}) = i \int_{\bar{z}}^{z} \log\left(\frac{2(1+3\zeta^2)}{(1-\zeta)(1+\zeta)^2}\right) \frac{d\zeta}{\zeta^2-1},$$

where z, $\Im(z) > 0$ is a root of the cubic equation $z^3 + 7z^2 - z + 1 = 0$.

By A. Mednykh and A. Vesnin (1998), the three-fold covering of $5_2(\frac{2\pi}{3})$ is the Weeks–Fomenko–Matveev manifold $\mathcal{M}_1$, which is the unique smallest volume closed hyperbolic 3-manifold. Its volume $\mathrm{Vol}\,\mathcal{M}_1 = 0.9427073627$ is well-known. Hence,

$$\mathrm{Vol}\, 5_2(\frac{2\pi}{5}) = \frac{1}{3}\mathrm{Vol}\,\mathcal{M}_1 = 0.3142357876,$$

which numerically coincides with the above formula.

Consider a twist link with rational slope $(4p+4)/(2p+1)$.
A particular element of this family for $p=1$ is the Whitehead link 5_1^2.
For $p=2,3$ the respective links are 6_3^2 and 7_3^2 in Rolfsen's notation.

We study cone manifold $7_2^3(\alpha,\beta)$ whose singular set is the link 7_3^2 with conical angles α and β along the singular components.

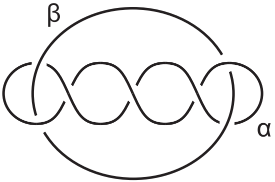

Fig.: Cone manifold $7_2^3(\alpha,\beta)$

Theorem (A. and Mednykh)

Let $7_3^2(\alpha,\beta)$ *be a hyperbolic cone manifold. Then the volume of* $7_3^2(\alpha,\beta)$ *is given by the formula* $\mathrm{Vol}\, 7_3^2(\alpha,\beta) =$

$$i\int_{\bar{z}}^{z} \log \frac{4(t^2+A^2)(t^2+B^2)}{(1+A^2)(1+B^2)t^2(2+t+t^2+(t-1)\sqrt{2+2t+t^2})}\,\frac{dt}{t^2-1},$$

where $A = \cot\frac{\alpha}{2}, B = \cot\frac{\beta}{2}$ *and* $z, \Im(z) > 0$ *is a root of the integrand.*

A background for the proof of this theorem was prepared in the paper by D. Derevnin, A. Mednykh and M. Mulazzani (2004). It is based on the Tangent rule for a general twist link with rational slope $(4p+4)/(2p+1)$ and the Schläfli formula.

The discussed results are presented in our paper:

N. Abrosimov and A. Mednykh, *Geometry of knots and links* // accepted for publication in special volume dedicated to V. Turaev, European Math. Society. monograph series

Thank you for your attention!