

Polyhedral products, graph products and p -central series

Taras Panov

Moscow State University

Groups and Quandles in Low-Dimensional Topology
Tomsk, 10–11 June 2025

The work is supported by the Russian Science Foundation,
grant no. 23-11-00143

Polyhedral product

$(\mathbf{X}, \mathbf{A}) = \{(X_1, A_1), \dots, (X_m, A_m)\}$ a sequence of pairs of spaces, $A_i \subset X_i$.

\mathcal{K} a simplicial complex on $[m] = \{1, 2, \dots, m\}$, $\emptyset \in \mathcal{K}$.

Given $I = \{i_1, \dots, i_k\} \subset [m]$, set

$$(\mathbf{X}, \mathbf{A})^I = Y_1 \times \dots \times Y_m \quad \text{where } Y_i = \begin{cases} X_i & \text{if } i \in I, \\ A_i & \text{if } i \notin I. \end{cases}$$

The **polyhedral product** of (\mathbf{X}, \mathbf{A}) over \mathcal{K} is

$$(\mathbf{X}, \mathbf{A})^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\mathbf{X}, \mathbf{A})^I = \bigcup_{I \in \mathcal{K}} \left(\prod_{i \in I} X_i \times \prod_{j \notin I} A_j \right) \subset \prod_{i=1}^m X_i.$$

Polyhedral product

$(\mathbf{X}, \mathbf{A}) = \{(X_1, A_1), \dots, (X_m, A_m)\}$ a sequence of pairs of spaces, $A_i \subset X_i$.

\mathcal{K} a simplicial complex on $[m] = \{1, 2, \dots, m\}$, $\emptyset \in \mathcal{K}$.

Given $I = \{i_1, \dots, i_k\} \subset [m]$, set

$$(\mathbf{X}, \mathbf{A})^I = Y_1 \times \dots \times Y_m \quad \text{where } Y_i = \begin{cases} X_i & \text{if } i \in I, \\ A_i & \text{if } i \notin I. \end{cases}$$

The **polyhedral product** of (\mathbf{X}, \mathbf{A}) over \mathcal{K} is

$$(\mathbf{X}, \mathbf{A})^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\mathbf{X}, \mathbf{A})^I = \bigcup_{I \in \mathcal{K}} \left(\prod_{i \in I} X_i \times \prod_{j \notin I} A_j \right) \subset \prod_{i=1}^m X_i.$$

Notation: $(X, A)^{\mathcal{K}} = (\mathbf{X}, \mathbf{A})^{\mathcal{K}}$ when all $(X_i, A_i) = (X, A)$;

$\mathbf{X}^{\mathcal{K}} = (\mathbf{X}, pt)^{\mathcal{K}}$, $X^{\mathcal{K}} = (X, pt)^{\mathcal{K}}$.

Category of faces $\text{CAT}(\mathcal{K})$.

Objects: simplices $I \in \mathcal{K}$. Morphisms: inclusions $I \subset J$.

Category of faces $\text{CAT}(\mathcal{K})$.

Objects: simplices $I \in \mathcal{K}$. Morphisms: inclusions $I \subset J$.

TOP the category of topological spaces.

Define the $\text{CAT}(\mathcal{K})$ -diagram

$$\begin{aligned} \mathcal{T}_{\mathcal{K}}(\mathbf{X}, \mathbf{A}): \text{CAT}(\mathcal{K}) &\longrightarrow \text{TOP}, \\ I &\longmapsto (\mathbf{X}, \mathbf{A})^I, \end{aligned}$$

which maps the morphism $I \subset J$ of $\text{CAT}(\mathcal{K})$ to the inclusion of spaces $(\mathbf{X}, \mathbf{A})^I \subset (\mathbf{X}, \mathbf{A})^J$.

Category of faces $\text{CAT}(\mathcal{K})$.

Objects: simplices $I \in \mathcal{K}$. Morphisms: inclusions $I \subset J$.

TOP the category of topological spaces.

Define the $\text{CAT}(\mathcal{K})$ -diagram

$$\begin{aligned} \mathcal{T}_{\mathcal{K}}(\mathbf{X}, \mathbf{A}): \text{CAT}(\mathcal{K}) &\longrightarrow \text{TOP}, \\ I &\longmapsto (\mathbf{X}, \mathbf{A})^I, \end{aligned}$$

which maps the morphism $I \subset J$ of $\text{CAT}(\mathcal{K})$ to the inclusion of spaces $(\mathbf{X}, \mathbf{A})^I \subset (\mathbf{X}, \mathbf{A})^J$.

Then

$$(\mathbf{X}, \mathbf{A})^{\mathcal{K}} = \text{colim}^{\text{TOP}} \mathcal{T}_{\mathcal{K}}(\mathbf{X}, \mathbf{A}) = \text{colim}_{I \in \mathcal{K}}^{\text{TOP}} (\mathbf{X}, \mathbf{A})^I.$$

○ a symmetric monoidal category with finite colimits.

\mathcal{C} a symmetric monoidal category with finite colimits.

$\mathbf{X} = (X_1, \dots, X_m)$ a sequence of objects in \mathcal{C} .

\mathcal{C} a symmetric monoidal category with finite colimits.

$\mathbf{X} = (X_1, \dots, X_m)$ a sequence of objects in \mathcal{C} . Set

$$\mathbf{X}^I = \prod_{i \in I} X_i,$$

where the product is taken with respect to the monoidal structure.

\mathcal{C} a symmetric monoidal category with finite colimits.

$\mathbf{X} = (X_1, \dots, X_m)$ a sequence of objects in \mathcal{C} . Set

$$\mathbf{X}^I = \prod_{i \in I} X_i,$$

where the product is taken with respect to the monoidal structure.

Consider the diagram

$$\mathcal{C}_{\mathcal{K}}(\mathbf{X}): \text{CAT}(\mathcal{K}) \longrightarrow \mathcal{C}, \quad I \longmapsto \mathbf{X}^I,$$

taking a morphism $I \subset J$ to the canonical morphism $\mathbf{X}^I \rightarrow \mathbf{X}^J$.

\mathcal{C} a symmetric monoidal category with finite colimits.

$\mathbf{X} = (X_1, \dots, X_m)$ a sequence of objects in \mathcal{C} . Set

$$\mathbf{X}^I = \prod_{i \in I} X_i,$$

where the product is taken with respect to the monoidal structure.

Consider the diagram

$$\mathcal{C}_{\mathcal{K}}(\mathbf{X}): \text{CAT}(\mathcal{K}) \longrightarrow \mathcal{C}, \quad I \longmapsto \mathbf{X}^I,$$

taking a morphism $I \subset J$ to the canonical morphism $\mathbf{X}^I \rightarrow \mathbf{X}^J$.

The **polyhedral product**

$$\mathbf{X}^{\mathcal{K}} = \text{colim}^{\mathcal{C}} \mathcal{C}_{\mathcal{K}}(\mathbf{X}) = \text{colim}_{I \in \mathcal{K}}^{\mathcal{C}} \mathbf{X}^I.$$

$\partial\Delta_{[m]}$: boundary of simplex on $[m]$.

The polyhedral product $\mathbf{X}^{\partial\Delta_{[m]}}$ of pointed topological spaces X_1, \dots, X_m is known as their **fat wedge**. When $m = 2$, it is the wedge $X_1 \vee X_2$.

$\partial\Delta_{[m]}$: boundary of simplex on $[m]$.

The polyhedral product $\mathbf{X}^{\partial\Delta_{[m]}}$ of pointed topological spaces X_1, \dots, X_m is known as their **fat wedge**. When $m = 2$, it is the wedge $X_1 \vee X_2$.

Definition

A symmetric monoidal category \mathcal{C} with finite colimits is **fat** if the canonical morphism

$$\mathbf{X}^{\partial\Delta_{[m]}} \rightarrow \mathbf{X}^{\Delta_{[m]}} = \prod_{i=1}^m X_i$$

is an isomorphism for any objects X_1, \dots, X_m and $m \geq 3$.

Examples of fat categories:

GRP: groups (the monoidal product is cartesian, the coproduct is the free product);

TMN: topological monoids (the monoidal product is cartesian, the coproduct is the free product);

TGP: topological groups (a full subcategory of **TMN**);

ASS: associative algebras with unit over a commutative ring R (the monoidal product is the tensor product, the coproduct is the free product);

LIE: Lie algebras over R (the monoidal product is the direct sum, the coproduct is the free product).

Examples of fat categories:

GRP: groups (the monoidal product is cartesian, the coproduct is the free product);

TMN: topological monoids (the monoidal product is cartesian, the coproduct is the free product);

TGP: topological groups (a full subcategory of **TMN**);

ASS: associative algebras with unit over a commutative ring R (the monoidal product is the tensor product, the coproduct is the free product);

LIE: Lie algebras over R (the monoidal product is the direct sum, the coproduct is the free product).

In each of these categories, a set of pairwise commuting elements of a group (algebra) generates a commutative subgroup (subalgebra).

Examples of fat categories:

GRP: groups (the monoidal product is cartesian, the coproduct is the free product);

TMN: topological monoids (the monoidal product is cartesian, the coproduct is the free product);

TGP: topological groups (a full subcategory of **TMN**);

ASS: associative algebras with unit over a commutative ring R (the monoidal product is the tensor product, the coproduct is the free product);

LIE: Lie algebras over R (the monoidal product is the direct sum, the coproduct is the free product).

In each of these categories, a set of pairwise commuting elements of a group (algebra) generates a commutative subgroup (subalgebra).

On the other hand, the category **TOP** is clearly not fat.

Proposition

In a fat category \mathcal{C} , the morphism $\mathbf{X}^{\mathcal{K}^1} \rightarrow \mathbf{X}^{\mathcal{K}}$ is an isomorphism for any simplicial complex \mathcal{K} and sequence of objects $\mathbf{X} = (X_1, \dots, X_m)$.

Proposition

In a fat category \mathcal{C} , the morphism $\mathbf{X}^{\mathcal{K}^1} \rightarrow \mathbf{X}^{\mathcal{K}}$ is an isomorphism for any simplicial complex \mathcal{K} and sequence of objects $\mathbf{X} = (X_1, \dots, X_m)$.

Γ a simple graph.

Complete subgraphs in Γ form a flag simplicial complex $\mathcal{K}(\Gamma)$, the **clique complex** of Γ .

Every flag complex \mathcal{K} is the clique complex of its 1-skeleton \mathcal{K}^1 .

Proposition

In a fat category \mathcal{C} , the morphism $\mathbf{X}^{\mathcal{K}^1} \rightarrow \mathbf{X}^{\mathcal{K}}$ is an isomorphism for any simplicial complex \mathcal{K} and sequence of objects $\mathbf{X} = (X_1, \dots, X_m)$.

Γ a simple graph.

Complete subgraphs in Γ form a flag simplicial complex $\mathcal{K}(\Gamma)$, the **clique complex** of Γ .

Every flag complex \mathcal{K} is the clique complex of its 1-skeleton \mathcal{K}^1 .

For a sequence $\mathbf{X} = (X_1, \dots, X_m)$ of objects in \mathcal{C} and a simple graph Γ on m vertices, the **graph product** \mathbf{X}^Γ is the polyhedral product $\mathbf{X}^{\mathcal{K}(\Gamma)}$.

Proposition

In a fat category \mathcal{C} , the morphism $\mathbf{X}^{\mathcal{K}^1} \rightarrow \mathbf{X}^{\mathcal{K}}$ is an isomorphism for any simplicial complex \mathcal{K} and sequence of objects $\mathbf{X} = (X_1, \dots, X_m)$.

Γ a simple graph.

Complete subgraphs in Γ form a flag simplicial complex $\mathcal{K}(\Gamma)$, the **clique complex** of Γ .

Every flag complex \mathcal{K} is the clique complex of its 1-skeleton \mathcal{K}^1 .

For a sequence $\mathbf{X} = (X_1, \dots, X_m)$ of objects in \mathcal{C} and a simple graph Γ on m vertices, the **graph product** \mathbf{X}^Γ is the polyhedral product $\mathbf{X}^{\mathcal{K}(\Gamma)}$.

If \mathcal{C} is fat, then \mathbf{X}^Γ is canonically isomorphic to $\mathbf{X}^{\mathcal{K}}$ for any \mathcal{K} with $\mathcal{K}^1 = \Gamma$.

Example (graph product of groups)

$\mathbf{G} = (G_1, \dots, G_m)$ a sequence of m (topological) groups.

Example (graph product of groups)

$\mathbf{G} = (G_1, \dots, G_m)$ a sequence of m (topological) groups.

Since the category TGP is fat, we have

$$\mathbf{G}^{\mathcal{K}^1} = \mathbf{G}^{\mathcal{K}} = \text{colim}_{I \in \mathcal{K}}^{\text{TGP}} \mathbf{G}^I.$$

Example (graph product of groups)

$\mathbf{G} = (G_1, \dots, G_m)$ a sequence of m (topological) groups.

Since the category TGP is fat, we have

$$\mathbf{G}^{\mathcal{K}^1} = \mathbf{G}^{\mathcal{K}} = \text{colim}_{I \in \mathcal{K}}^{\text{TGP}} \mathbf{G}^I.$$

Explicitly,

$$\mathbf{G}^{\mathcal{K}} = \bigstar_{k=1}^m G_k / (g_i g_j = g_j g_i \text{ for } g_i \in G_i, g_j \in G_j, \{i, j\} \in \mathcal{K}),$$

where $\bigstar_{k=1}^m G_k$ denotes the free product of the groups G_k .

Example (graph product of groups)

$\mathbf{G} = (G_1, \dots, G_m)$ a sequence of m (topological) groups.

Since the category TGP is fat, we have

$$\mathbf{G}^{\mathcal{K}^1} = \mathbf{G}^{\mathcal{K}} = \text{colim}_{I \in \mathcal{K}}^{\text{TGP}} \mathbf{G}^I.$$

Explicitly,

$$\mathbf{G}^{\mathcal{K}} = \bigstar_{k=1}^m G_k / (g_i g_j = g_j g_i \text{ for } g_i \in G_i, g_j \in G_j, \{i, j\} \in \mathcal{K}),$$

where $\bigstar_{k=1}^m G_k$ denotes the free product of the groups G_k .

There are canonical injective homomorphisms $\mathbf{G}^I \rightarrow \mathbf{G}^{\mathcal{K}}$ for $I \in \mathcal{K}$.

When a functor between two categories preserves graph products? This is certainly the case when the functor preserves finite products and colimits. Other examples include the classifying space functor and the loop homology functor, considered next.

When a functor between two categories preserves graph products? This is certainly the case when the functor preserves finite products and colimits. Other examples include the classifying space functor and the loop homology functor, considered next.

BG the classifying space for G , $EG \rightarrow BG$ the universal G -bundle.

When a functor between two categories preserves graph products? This is certainly the case when the functor preserves finite products and colimits. Other examples include the classifying space functor and the loop homology functor, considered next.

BG the classifying space for G , $EG \rightarrow BG$ the universal G -bundle.

$$B\mathbf{G}^I = \prod_{i \in I} BG_i.$$

$(B\mathbf{G})^{\mathcal{K}}$ the polyhedral product corresponding to the sequence of pairs $(B\mathbf{G}, pt) = \{(BG_1, pt), \dots, (BG_m, pt)\}$ in TOP .

When a functor between two categories preserves graph products? This is certainly the case when the functor preserves finite products and colimits. Other examples include the classifying space functor and the loop homology functor, considered next.

BG the classifying space for G , $EG \rightarrow BG$ the universal G -bundle.

$$B\mathbf{G}^I = \prod_{i \in I} BG_i.$$

$(B\mathbf{G})^{\mathcal{K}}$ the polyhedral product corresponding to the sequence of pairs

$$(B\mathbf{G}, pt) = \{(BG_1, pt), \dots, (BG_m, pt)\} \text{ in TOP.}$$

$$(E\mathbf{G}, \mathbf{G})^{\mathcal{K}} \text{ for } (E\mathbf{G}, \mathbf{G}) = \{(EG_1, G_1), \dots, (EG_m, G_m)\}.$$

Here each G_i is included in EG_i as the fibre of $EG_i \rightarrow BG_i$ over pt .

When a functor between two categories preserves graph products? This is certainly the case when the functor preserves finite products and colimits. Other examples include the classifying space functor and the loop homology functor, considered next.

BG the classifying space for G , $EG \rightarrow BG$ the universal G -bundle.

$$B\mathbf{G}^I = \prod_{i \in I} BG_i.$$

$(B\mathbf{G})^{\mathcal{K}}$ the polyhedral product corresponding to the sequence of pairs $(B\mathbf{G}, pt) = \{(BG_1, pt), \dots, (BG_m, pt)\}$ in TOP.

$(E\mathbf{G}, \mathbf{G})^{\mathcal{K}}$ for $(E\mathbf{G}, \mathbf{G}) = \{(EG_1, G_1), \dots, (EG_m, G_m)\}$.

Here each G_i is included in EG_i as the fibre of $EG_i \rightarrow BG_i$ over pt .

There is a homotopy fibration of polyhedral products

$$(E\mathbf{G}, \mathbf{G})^{\mathcal{K}} \longrightarrow (B\mathbf{G})^{\mathcal{K}} \longrightarrow \prod_{i=1}^m BG_i.$$

The classifying space functor $B: \mathbf{TGP} \rightarrow \mathbf{TOP}$ does not preserve colimits over small categories in general, but it does preserve *homotopy colimits* under appropriate model structures on \mathbf{TOP} and \mathbf{TGP} .

The classifying space functor $B: \text{TGP} \rightarrow \text{TOP}$ does not preserve colimits over small categories in general, but it does preserve *homotopy colimits* under appropriate model structures on TOP and TGP .

Theorem (P-Ray-Vogt)

There is a commutative square of natural maps

$$\begin{array}{ccc}
 \text{hocolim}_{I \in \mathcal{K}}^{\text{TOP}} B\mathbf{G}^I & \xrightarrow{\simeq} & B \text{hocolim}_{I \in \mathcal{K}}^{\text{TGP}} \mathbf{G}^I \\
 \downarrow \simeq & & \downarrow \\
 \text{colim}_{I \in \mathcal{K}}^{\text{TOP}} B\mathbf{G}^I & \longrightarrow & B \text{colim}_{I \in \mathcal{K}}^{\text{TGP}} \mathbf{G}^I,
 \end{array}$$

in TOP , where the top and left maps are homotopy equivalences, and the right map is a homotopy equivalence when \mathcal{K} is a flag complex.

The classifying space functor $B: \text{TGP} \rightarrow \text{TOP}$ does not preserve colimits over small categories in general, but it does preserve *homotopy colimits* under appropriate model structures on TOP and TGP .

Theorem (P-Ray-Vogt)

There is a commutative square of natural maps

$$\begin{array}{ccc}
 \text{hocolim}_{I \in \mathcal{K}}^{\text{TOP}} B\mathbf{G}^I & \xrightarrow{\simeq} & B \text{hocolim}_{I \in \mathcal{K}}^{\text{TGP}} \mathbf{G}^I \\
 \downarrow \simeq & & \downarrow \\
 \text{colim}_{I \in \mathcal{K}}^{\text{TOP}} B\mathbf{G}^I & \longrightarrow & B \text{colim}_{I \in \mathcal{K}}^{\text{TGP}} \mathbf{G}^I,
 \end{array}$$

in TOP , where the top and left maps are homotopy equivalences, and the right map is a homotopy equivalence when \mathcal{K} is a flag complex.

Corollary

$(B\mathbf{G})^{\mathcal{K}} \rightarrow B(\mathbf{G}^{\mathcal{K}})$ is a homotopy equivalence when \mathcal{K} is flag.

The classifying space functor $B: \text{TGP} \rightarrow \text{TOP}$ does not preserve colimits over small categories in general, but it does preserve *homotopy colimits* under appropriate model structures on TOP and TGP .

Theorem (P-Ray-Vogt)

There is a commutative square of natural maps

$$\begin{array}{ccc}
 \text{hocolim}_{I \in \mathcal{K}}^{\text{TOP}} B\mathbf{G}^I & \xrightarrow{\simeq} & B \text{hocolim}_{I \in \mathcal{K}}^{\text{TGP}} \mathbf{G}^I \\
 \downarrow \simeq & & \downarrow \\
 \text{colim}_{I \in \mathcal{K}}^{\text{TOP}} B\mathbf{G}^I & \longrightarrow & B \text{colim}_{I \in \mathcal{K}}^{\text{TGP}} \mathbf{G}^I,
 \end{array}$$

in TOP , where the top and left maps are homotopy equivalences, and the right map is a homotopy equivalence when \mathcal{K} is a flag complex.

Corollary

$(B\mathbf{G})^{\mathcal{K}} \rightarrow B(\mathbf{G}^{\mathcal{K}})$ is a homotopy equivalence when \mathcal{K} is flag. That is, the classifying space functor preserves graph products.

Example

Let \mathcal{K} be a flag complex in these examples.

Example

Let \mathcal{K} be a flag complex in these examples.

1. Let $G_i = \mathbb{Z}$. Then $\mathbf{G}^{\mathcal{K}}$ is the **right-angled Artin group**

$$RA_{\mathcal{K}} = \mathbb{Z}^{\mathcal{K}} = F(a_1, \dots, a_m) / (a_i a_j = a_j a_i \text{ for } \{i, j\} \in \mathcal{K}),$$

where $F(a_1, \dots, a_m)$ is a free group on m generators.

Example

Let \mathcal{K} be a flag complex in these examples.

1. Let $G_i = \mathbb{Z}$. Then $\mathbf{G}^{\mathcal{K}}$ is the **right-angled Artin group**

$$RA_{\mathcal{K}} = \mathbb{Z}^{\mathcal{K}} = F(a_1, \dots, a_m) / (a_i a_j = a_j a_i \text{ for } \{i, j\} \in \mathcal{K}),$$

where $F(a_1, \dots, a_m)$ is a free group on m generators. The classifying space of $RA_{\mathcal{K}}$ is the polyhedral product $(B\mathbb{Z})^{\mathcal{K}} = (S^1)^{\mathcal{K}}$, a subcomplex of the m -torus $T^m = (S^1)^m$.

Example

Let \mathcal{K} be a flag complex in these examples.

1. Let $G_i = \mathbb{Z}$. Then $\mathbf{G}^{\mathcal{K}}$ is the **right-angled Artin group**

$$RA_{\mathcal{K}} = \mathbb{Z}^{\mathcal{K}} = F(a_1, \dots, a_m) / (a_i a_j = a_j a_i \text{ for } \{i, j\} \in \mathcal{K}),$$

where $F(a_1, \dots, a_m)$ is a free group on m generators. The classifying space of $RA_{\mathcal{K}}$ is the polyhedral product $(B\mathbb{Z})^{\mathcal{K}} = (S^1)^{\mathcal{K}}$, a subcomplex of the m -torus $T^m = (S^1)^m$. There is a homotopy fibration

$$(\mathbb{R}, \mathbb{Z})^{\mathcal{K}} \longrightarrow (S^1)^{\mathcal{K}} \longrightarrow (S^1)^m,$$

where all spaces are aspherical.

Example

Let \mathcal{K} be a flag complex in these examples.

1. Let $G_i = \mathbb{Z}$. Then $\mathbf{G}^{\mathcal{K}}$ is the **right-angled Artin group**

$$RA_{\mathcal{K}} = \mathbb{Z}^{\mathcal{K}} = F(a_1, \dots, a_m) / (a_i a_j = a_j a_i \text{ for } \{i, j\} \in \mathcal{K}),$$

where $F(a_1, \dots, a_m)$ is a free group on m generators. The classifying space of $RA_{\mathcal{K}}$ is the polyhedral product $(B\mathbb{Z})^{\mathcal{K}} = (S^1)^{\mathcal{K}}$, a subcomplex of the m -torus $T^m = (S^1)^m$. There is a homotopy fibration

$$(\mathbb{R}, \mathbb{Z})^{\mathcal{K}} \longrightarrow (S^1)^{\mathcal{K}} \longrightarrow (S^1)^m,$$

where all spaces are aspherical. Passing to the fundamental groups,

$$1 \longrightarrow RA'_{\mathcal{K}} \longrightarrow RA_{\mathcal{K}} \xrightarrow{Ab} \mathbb{Z}^m \longrightarrow 1,$$

where Ab is the abelianisation map and $RA'_{\mathcal{K}}$ is the commutator subgroup.

Example

2. Let $G_i = \mathbb{Z}_2$, a cyclic group of order 2, for $i = 1, \dots, m$.

Then $\mathbf{G}^{\mathcal{K}}$ is the **right-angled Coxeter group**

$$RC_{\mathcal{K}} = \mathbb{Z}_2^{\mathcal{K}} = F(a_1, \dots, a_m) / (a_i^2 = 1, a_i a_j = a_j a_i \text{ for } \{i, j\} \in \mathcal{K}).$$

Example

2. Let $G_i = \mathbb{Z}_2$, a cyclic group of order 2, for $i = 1, \dots, m$.

Then $\mathbf{G}^{\mathcal{K}}$ is the **right-angled Coxeter group**

$$RC_{\mathcal{K}} = \mathbb{Z}_2^{\mathcal{K}} = F(a_1, \dots, a_m) / (a_i^2 = 1, a_i a_j = a_j a_i \text{ for } \{i, j\} \in \mathcal{K}).$$

The classifying space of $RC_{\mathcal{K}}$ is $(B\mathbb{Z}_2)^{\mathcal{K}} = (\mathbb{R}P^\infty)^{\mathcal{K}}$.

The polyhedral product $(EZ_2, \mathbb{Z}_2)^{\mathcal{K}}$ is homotopy equivalent to the **real moment-angle complex** $\mathcal{R}_{\mathcal{K}} = (D^1, S^0)^{\mathcal{K}}$.

Example

2. Let $G_i = \mathbb{Z}_2$, a cyclic group of order 2, for $i = 1, \dots, m$.

Then $\mathbf{G}^{\mathcal{K}}$ is the **right-angled Coxeter group**

$$RC_{\mathcal{K}} = \mathbb{Z}_2^{\mathcal{K}} = F(a_1, \dots, a_m) / (a_i^2 = 1, a_i a_j = a_j a_i \text{ for } \{i, j\} \in \mathcal{K}).$$

The classifying space of $RC_{\mathcal{K}}$ is $(B\mathbb{Z}_2)^{\mathcal{K}} = (\mathbb{R}P^\infty)^{\mathcal{K}}$.

The polyhedral product $(E\mathbb{Z}_2, \mathbb{Z}_2)^{\mathcal{K}}$ is homotopy equivalent to the **real moment-angle complex** $\mathcal{R}_{\mathcal{K}} = (D^1, S^0)^{\mathcal{K}}$. There is a homotopy fibration

$$\mathcal{R}_{\mathcal{K}} \longrightarrow (\mathbb{R}P^\infty)^{\mathcal{K}} \longrightarrow (\mathbb{R}P^\infty)^m,$$

where all spaces are aspherical.

Example

2. Let $G_i = \mathbb{Z}_2$, a cyclic group of order 2, for $i = 1, \dots, m$.

Then $\mathbf{G}^{\mathcal{K}}$ is the **right-angled Coxeter group**

$$RC_{\mathcal{K}} = \mathbb{Z}_2^{\mathcal{K}} = F(a_1, \dots, a_m) / (a_i^2 = 1, a_i a_j = a_j a_i \text{ for } \{i, j\} \in \mathcal{K}).$$

The classifying space of $RC_{\mathcal{K}}$ is $(B\mathbb{Z}_2)^{\mathcal{K}} = (\mathbb{R}P^\infty)^{\mathcal{K}}$.

The polyhedral product $(E\mathbb{Z}_2, \mathbb{Z}_2)^{\mathcal{K}}$ is homotopy equivalent to the **real moment-angle complex** $\mathcal{R}_{\mathcal{K}} = (D^1, S^0)^{\mathcal{K}}$. There is a homotopy fibration

$$\mathcal{R}_{\mathcal{K}} \longrightarrow (\mathbb{R}P^\infty)^{\mathcal{K}} \longrightarrow (\mathbb{R}P^\infty)^m,$$

where all spaces are aspherical. Passing to the fundamental groups,

$$1 \longrightarrow RC'_{\mathcal{K}} \longrightarrow RC_{\mathcal{K}} \xrightarrow{Ab} \mathbb{Z}_2^m \longrightarrow 1.$$

Example

3. Let $G_i = T^1$ (a circle group) for $i = 1, \dots, m$. The corresponding graph product $\mathbf{G}^{\mathcal{K}}$ is the colimit of tori T^I , known as the **circulation group**:

$$\text{Cir}_{\mathcal{K}} = T^{\mathcal{K}} = \text{colim}_{I \in \mathcal{K}}^{\text{TGP}} T^I.$$

Example

3. Let $G_i = T^1$ (a circle group) for $i = 1, \dots, m$. The corresponding graph product $\mathbf{G}^{\mathcal{K}}$ is the colimit of tori T^I , known as the **circulation group**:

$$\text{Cir}_{\mathcal{K}} = T^{\mathcal{K}} = \text{colim}_{I \in \mathcal{K}}^{\text{TGP}} T^I.$$

Its classifying space is $(BT^1)^{\mathcal{K}} = (\mathbb{C}P^\infty)^{\mathcal{K}}$, the **Davis–Januszkiewicz space**. The polyhedral product $(ET^1, T^1)^{\mathcal{K}}$ is homotopy equivalent to the **moment-angle complex** $\mathcal{Z}_{\mathcal{K}} = (D^2, S^1)^{\mathcal{K}}$. There is a homotopy fibration

$$\mathcal{Z}_{\mathcal{K}} \longrightarrow (\mathbb{C}P^\infty)^{\mathcal{K}} \longrightarrow (\mathbb{C}P^\infty)^m.$$

Example

3. Let $G_i = T^1$ (a circle group) for $i = 1, \dots, m$. The corresponding graph product $\mathbf{G}^{\mathcal{K}}$ is the colimit of tori T^I , known as the **circulation group**:

$$\text{Cir}_{\mathcal{K}} = T^{\mathcal{K}} = \text{colim}_{I \in \mathcal{K}}^{\text{TGP}} T^I.$$

Its classifying space is $(BT^1)^{\mathcal{K}} = (\mathbb{C}P^\infty)^{\mathcal{K}}$, the **Davis–Januszkiewicz space**. The polyhedral product $(ET^1, T^1)^{\mathcal{K}}$ is homotopy equivalent to the **moment-angle complex** $\mathcal{Z}_{\mathcal{K}} = (D^2, S^1)^{\mathcal{K}}$. There is a homotopy fibration

$$\mathcal{Z}_{\mathcal{K}} \longrightarrow (\mathbb{C}P^\infty)^{\mathcal{K}} \longrightarrow (\mathbb{C}P^\infty)^m.$$

Applying the Moore loop space functor $\Omega: \text{TOP} \rightarrow \text{TMN}$ we obtain a commutative diagram in $\text{Ho}(\text{TMN})$

$$\begin{array}{ccccccc} \Omega \mathcal{Z}_{\mathcal{K}} & \longrightarrow & \Omega(\mathbb{C}P^\infty)^{\mathcal{K}} & \longrightarrow & T^m & & \\ \downarrow \simeq & & \downarrow \simeq & & \parallel & & \\ 1 & \longrightarrow & \text{Cir}'_{\mathcal{K}} & \longrightarrow & \text{Cir}_{\mathcal{K}} & \xrightarrow{\text{Ab}} & T^m \longrightarrow 1 \end{array}$$

Example

4. More generally, let G_i be the Moore loop monoid ΩX_i of a simply connected space X_i , $i = 1, \dots, m$.

Example

4. More generally, let G_i be the Moore loop monoid ΩX_i of a simply connected space X_i , $i = 1, \dots, m$. There is an equivalence in $Ho(\text{TMN})$

$$\Omega(\mathbf{X}^{\mathcal{K}}) \simeq (\Omega \mathbf{X})^{\mathcal{K}} = \text{colim}_{I \in \mathcal{K}}^{\text{TMN}} (\Omega \mathbf{X})^I$$

for flag \mathcal{K} . The Moore loop functor preserves graph products up to htopy.

Example

4. More generally, let G_i be the Moore loop monoid ΩX_i of a simply connected space X_i , $i = 1, \dots, m$. There is an equivalence in $Ho(\text{TMN})$

$$\Omega(\mathbf{X}^{\mathcal{K}}) \simeq (\Omega \mathbf{X})^{\mathcal{K}} = \text{colim}_{I \in \mathcal{K}}^{\text{TMN}} (\Omega \mathbf{X})^I$$

for flag \mathcal{K} . The Moore loop functor preserves graph products up to htopy.

R a commutative ring R with unit.

The loop homology $H_*(\Omega X; R)$ of a simply connected space X is an associative noncommutative algebra with respect to the Pontryagin product.

Get a functor $H_*\Omega: \text{TOP} \rightarrow \text{ASS}$.

Example

4. More generally, let G_i be the Moore loop monoid ΩX_i of a simply connected space X_i , $i = 1, \dots, m$. There is an equivalence in $Ho(\text{TMN})$

$$\Omega(\mathbf{X}^{\mathcal{K}}) \simeq (\Omega \mathbf{X})^{\mathcal{K}} = \text{colim}_{I \in \mathcal{K}}^{\text{TMN}} (\Omega \mathbf{X})^I$$

for flag \mathcal{K} . The Moore loop functor preserves graph products up to htopy.

R a commutative ring R with unit.

The loop homology $H_*(\Omega X; R)$ of a simply connected space X is an associative noncommutative algebra with respect to the Pontryagin product.

Get a functor $H_*\Omega: \text{TOP} \rightarrow \text{ASS}$.

When R is a field or $H_*(\Omega X; R)$ is free over R , there is a cocommutative coproduct $H_*(\Omega X; R) \rightarrow H_*(\Omega X; R) \otimes H_*(\Omega X; R)$ induced by the diagonal map $\Delta: \Omega X \rightarrow \Omega X \times \Omega X$, which makes $H_*(\Omega X; R)$ into a cocommutative Hopf algebra.

Given a sequence $\mathbf{A} = (A_1, \dots, A_m)$ of associative algebras, the polyhedral product

$$\mathbf{A}^{\mathcal{K}} = \operatorname{colim}_{I \in \mathcal{K}}^{\operatorname{ASS}} \left(\bigotimes_{i \in I} A_i \right)$$

is also the graph product, since the category ASS is fat.

Given a sequence $\mathbf{A} = (A_1, \dots, A_m)$ of associative algebras, the polyhedral product

$$\mathbf{A}^{\mathcal{K}} = \operatorname{colim}_{I \in \mathcal{K}}^{\text{ASS}} \left(\bigotimes_{i \in I} A_i \right)$$

is also the graph product, since the category ASS is fat.

Theorem (Dobrinskaya, Cai)

Let \mathcal{K} be a flag complex, $\mathbf{X} = (X_1, \dots, X_m)$ a sequence of simply connected spaces, and R a field. There are isomorphisms

$$H_*(\Omega(\mathbf{X}^{\mathcal{K}}); R) \cong (H_*(\Omega\mathbf{X}; R))^{\mathcal{K}}$$

$$\cong \bigstar_{k=1}^m H_*(\Omega X_i) / ([x_i, x_j] = 0 \text{ for } x_i \in H_*(\Omega X_i), x_j \in H_*(\Omega X_j), \{i, j\} \in \mathcal{K})$$

of graded R -algebras, where $[x_i, x_j] = x_i x_j - (-1)^{|x_i||x_j|} x_j x_i$.

That is, the loop homology functor $H_*\Omega: \text{TOP} \rightarrow \text{ASS}$ with field coefficients preserves graph products.

The cases $X_i = S^{2p+1}$ and $X_i = \mathbb{C}P^\infty$ are of particular importance.

We have $H_*(\Omega S^{2p+1}; R) = R[v]$, $|v| = 2p$.

The cases $X_i = S^{2p+1}$ and $X_i = \mathbb{C}P^\infty$ are of particular importance.

We have $H_*(\Omega S^{2p+1}; R) = R[v]$, $|v| = 2p$.

For flag \mathcal{K} , the graph product algebra $H_*(\Omega(S^{2p+1})^{\mathcal{K}}; R)$ is the algebraic counterpart of the right-angled Artin group:

The cases $X_i = S^{2p+1}$ and $X_i = \mathbb{C}P^\infty$ are of particular importance.

We have $H_*(\Omega S^{2p+1}; R) = R[v]$, $|v| = 2p$.

For flag \mathcal{K} , the graph product algebra $H_*(\Omega(S^{2p+1})^{\mathcal{K}}; R)$ is the algebraic counterpart of the right-angled Artin group:

Theorem (Bubenik–Gold)

For any flag complex \mathcal{K} and $p \geq 1$, there are isomorphisms

$$\begin{aligned} H_*(\Omega(S^{2p+1})^{\mathcal{K}}; R) &\cong R[v]^{\mathcal{K}} = \operatorname{colim}_{I \in \mathcal{K}}^{\text{ASS}} R[v_i : i \in I] \\ &\cong T(v_1, \dots, v_m) / (v_i v_j - v_j v_i = 0 \text{ for } \{i, j\} \in \mathcal{K}) \end{aligned}$$

of graded R -algebras, where $T(v_1, \dots, v_m)$ is the free associative algebra, $R[v_i : i \in I]$ is the polynomial algebra and $|v_i| = 2p$.

Similarly, the algebra $H_*(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}}; R)$ is related to right-angled Coxeter groups. To elaborate on this, consider the exterior algebra

$$H_*(\Omega(\mathbb{C}P^\infty)^m; R) = H(T^m; R) = \Lambda[u_1, \dots, u_m],$$

where $|u_j| = 1$.

Similarly, the algebra $H_*(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}}; R)$ is related to right-angled Coxeter groups. To elaborate on this, consider the exterior algebra

$$H_*(\Omega(\mathbb{C}P^\infty)^m; R) = H(T^m; R) = \Lambda[u_1, \dots, u_m],$$

where $|u_j| = 1$. The homotopy fibration $\mathcal{Z}_{\mathcal{K}} \rightarrow (\mathbb{C}P^\infty)^{\mathcal{K}} \rightarrow (\mathbb{C}P^\infty)^m$ gives

$$R \longrightarrow H_*(\Omega\mathcal{Z}_{\mathcal{K}}; R) \longrightarrow H_*(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}}; R) \xrightarrow{Ab} \Lambda[u_1, \dots, u_m] \longrightarrow R.$$

Similarly, the algebra $H_*(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}}; R)$ is related to right-angled Coxeter groups. To elaborate on this, consider the exterior algebra

$$H_*(\Omega(\mathbb{C}P^\infty)^m; R) = H(T^m; R) = \Lambda[u_1, \dots, u_m],$$

where $|u_j| = 1$. The homotopy fibration $\mathcal{Z}_{\mathcal{K}} \rightarrow (\mathbb{C}P^\infty)^{\mathcal{K}} \rightarrow (\mathbb{C}P^\infty)^m$ gives

$$R \longrightarrow H_*(\Omega\mathcal{Z}_{\mathcal{K}}; R) \longrightarrow H_*(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}}; R) \xrightarrow{Ab} \Lambda[u_1, \dots, u_m] \longrightarrow R.$$

$H_*(\Omega\mathcal{Z}_{\mathcal{K}}; R)$ is the **commutator subalgebra** of $H_*(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}}; R)$.

Similarly, the algebra $H_*(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}}; R)$ is related to right-angled Coxeter groups. To elaborate on this, consider the exterior algebra

$$H_*(\Omega(\mathbb{C}P^\infty)^m; R) = H(T^m; R) = \Lambda[u_1, \dots, u_m],$$

where $|u_i| = 1$. The homotopy fibration $\mathcal{Z}_{\mathcal{K}} \rightarrow (\mathbb{C}P^\infty)^{\mathcal{K}} \rightarrow (\mathbb{C}P^\infty)^m$ gives

$$R \longrightarrow H_*(\Omega\mathcal{Z}_{\mathcal{K}}; R) \longrightarrow H_*(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}}; R) \xrightarrow{Ab} \Lambda[u_1, \dots, u_m] \longrightarrow R.$$

$H_*(\Omega\mathcal{Z}_{\mathcal{K}}; R)$ is the **commutator subalgebra** of $H_*(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}}; R)$.

Theorem (P-Ray)

For any flag complex \mathcal{K} , there are isomorphisms

$$\begin{aligned} H_*(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}}; R) &\cong \Lambda[u]_{\mathcal{K}} = \operatorname{colim}_{I \in \mathcal{K}}^{\text{ASS}} \Lambda[u_i : i \in I] \\ &\cong T(u_1, \dots, u_m) / (u_i^2 = 0, u_i u_j + u_j u_i = 0 \text{ for } \{i, j\} \in \mathcal{K}) \end{aligned}$$

of graded R -algebras, where $\Lambda[u_i : i \in I]$ is the exterior algebra and $|u_i| = 1$.

Central series and associated Lie algebras

Given two subgroups H and W of G , denote by (H, W) the subgroup generated by all commutators (h, w) with $h \in H$ and $w \in W$.
In particular, $(G, G) = G'$ is the commutator subgroup.

Central series and associated Lie algebras

Given two subgroups H and W of G , denote by (H, W) the subgroup generated by all commutators (h, w) with $h \in H$ and $w \in W$. In particular, $(G, G) = G'$ is the commutator subgroup.

A (descending) **central series** or **central filtration** on G is a sequence of subgroups $\Gamma(G) = \{\Gamma_k(G) : k \geq 1\}$ such that $\Gamma_1(G) = G$, $\Gamma_{k+1}(G) \subset \Gamma_k(G)$ and $(\Gamma_k(G), \Gamma_l(G)) \subset \Gamma_{k+l}(G)$ for any k, l .

Central series and associated Lie algebras

Given two subgroups H and W of G , denote by (H, W) the subgroup generated by all commutators (h, w) with $h \in H$ and $w \in W$. In particular, $(G, G) = G'$ is the commutator subgroup.

A (descending) **central series** or **central filtration** on G is a sequence of subgroups $\Gamma(G) = \{\Gamma_k(G) : k \geq 1\}$ such that $\Gamma_1(G) = G$, $\Gamma_{k+1}(G) \subset \Gamma_k(G)$ and $(\Gamma_k(G), \Gamma_l(G)) \subset \Gamma_{k+l}(G)$ for any k, l .

The fastest descending central series is the **lower central series** $\gamma(G)$, given by $\gamma_1(G) = G$ and $\gamma_k(G) = (\gamma_{k-1}(G), G)$ for $k \geq 2$.

Central series and associated Lie algebras

Given two subgroups H and W of G , denote by (H, W) the subgroup generated by all commutators (h, w) with $h \in H$ and $w \in W$. In particular, $(G, G) = G'$ is the commutator subgroup.

A (descending) **central series** or **central filtration** on G is a sequence of subgroups $\Gamma(G) = \{\Gamma_k(G) : k \geq 1\}$ such that $\Gamma_1(G) = G$, $\Gamma_{k+1}(G) \subset \Gamma_k(G)$ and $(\Gamma_k(G), \Gamma_l(G)) \subset \Gamma_{k+l}(G)$ for any k, l .

The fastest descending central series is the **lower central series** $\gamma(G)$, given by $\gamma_1(G) = G$ and $\gamma_k(G) = (\gamma_{k-1}(G), G)$ for $k \geq 2$.

The **associated Lie algebra** of a central series $\Gamma(G)$ is the associated graded abelian group

$$\text{gr}_\bullet \Gamma(G) = \bigoplus_{k \geq 1} \Gamma_k(G) / \Gamma_{k+1}(G)$$

with the Lie bracket defined by $[\bar{g}_k, \bar{g}_l] = \overline{(g_k, g_l)}$, where \bar{g}_k denotes the image of $g_k \in \Gamma_k(G)$ in the quotient group $\Gamma_k(G) / \Gamma_{k+1}(G)$.

By the classical result of Magnus, the Lie algebra associated with the lower central series of a free group $F(a_1, \dots, a_m)$ is a **free Lie algebra** $FL(\bar{a}_1, \dots, \bar{a}_m)$ over \mathbb{Z} .

By the classical result of Magnus, the Lie algebra associated with the lower central series of a free group $F(a_1, \dots, a_m)$ is a **free Lie algebra** $FL(\bar{a}_1, \dots, \bar{a}_m)$ over \mathbb{Z} . This generalises to right-angled Artin groups:

Theorem (Duchamp–Krob, Papadima–Suciu)

The Lie algebra associated with the lower central series of a right-angled Artin group $RA_{\mathcal{K}}$ is given by

$$\mathrm{gr}_{\bullet} \gamma(RA_{\mathcal{K}}) \cong FL(v_1, \dots, v_m) / ([v_i, v_j] = 0 \text{ for } \{i, j\} \in \mathcal{K}).$$

Furthermore, each component $\mathrm{gr}_k \gamma(RA_{\mathcal{K}})$ is a free abelian group.

By the classical result of Magnus, the Lie algebra associated with the lower central series of a free group $F(a_1, \dots, a_m)$ is a **free Lie algebra** $FL(\bar{a}_1, \dots, \bar{a}_m)$ over \mathbb{Z} . This generalises to right-angled Artin groups:

Theorem (Duchamp–Krob, Papadima–Suciu)

The Lie algebra associated with the lower central series of a right-angled Artin group $RA_{\mathcal{K}}$ is given by

$$\mathrm{gr}_{\bullet} \gamma(RA_{\mathcal{K}}) \cong FL(v_1, \dots, v_m) / ([v_i, v_j] = 0 \text{ for } \{i, j\} \in \mathcal{K}).$$

Furthermore, each component $\mathrm{gr}_k \gamma(RA_{\mathcal{K}})$ is a free abelian group.

The Lie algebra (over \mathbb{Z}) on the right hand side is known as the **graph Lie algebra**, or **partially commutative Lie algebra**. It is the graph product $\mathbb{Z}\langle v \rangle^{\mathcal{K}}$ of trivial Lie algebras $\mathbb{Z}\langle v \rangle = FL(v)$.

The universal enveloping functor $\mathcal{U}: \text{LIE} \rightarrow \text{ASS}$ is left adjoint and preserves products, so it also preserves graph products. We obtain

The universal enveloping functor $\mathcal{U}: \text{LIE} \rightarrow \text{ASS}$ is left adjoint and preserves products, so it also preserves graph products. We obtain

Proposition

For a flag complex \mathcal{K} , there is an isomorphism

$$\mathcal{U}(\text{gr}_{\bullet} \gamma(RA_{\mathcal{K}})) \cong H_*(\Omega(S^{2p+1})^{\mathcal{K}}; \mathbb{Z})$$

of associative \mathbb{Z} -algebras, where $p \geq 1$.

The universal enveloping functor $\mathcal{U}: \text{LIE} \rightarrow \text{ASS}$ is left adjoint and preserves products, so it also preserves graph products. We obtain

Proposition

For a flag complex \mathcal{K} , there is an isomorphism

$$\mathcal{U}(\text{gr}_{\bullet} \gamma(RA_{\mathcal{K}})) \cong H_*(\Omega(S^{2p+1})^{\mathcal{K}}; \mathbb{Z})$$

of associative \mathbb{Z} -algebras, where $p \geq 1$.

The lower central series of a right-angled Coxeter group $RC_{\mathcal{K}}$ is more subtle.

The universal enveloping functor $\mathcal{U}: \text{LIE} \rightarrow \text{ASS}$ is left adjoint and preserves products, so it also preserves graph products. We obtain

Proposition

For a flag complex \mathcal{K} , there is an isomorphism

$$\mathcal{U}(\text{gr}_\bullet \gamma(RA_{\mathcal{K}})) \cong H_*(\Omega(S^{2p+1})^{\mathcal{K}}; \mathbb{Z})$$

of associative \mathbb{Z} -algebras, where $p \geq 1$.

The lower central series of a right-angled Coxeter group $RC_{\mathcal{K}}$ is more subtle. One has $(\gamma_k(RC_{\mathcal{K}}))^2 \subset \gamma_{k+1}(RC_{\mathcal{K}})$, so $\text{gr}_\bullet \gamma(RC_{\mathcal{K}})$ is a Lie algebra over \mathbb{Z}_2 . There is an epimorphism

$$FL_{\mathbb{Z}_2}(v_1, \dots, v_m) / ([v_i, v_j] = 0 \text{ for } \{i, j\} \in \mathcal{K}) \rightarrow \text{gr}_\bullet \gamma(RC_{\mathcal{K}})$$

of Lie algebras over \mathbb{Z}_2 , where $FL_{\mathbb{Z}_2}(v_1, \dots, v_m) = FL(v_1, \dots, v_m) \otimes \mathbb{Z}_2$.

The universal enveloping functor $\mathcal{U}: \text{LIE} \rightarrow \text{ASS}$ is left adjoint and preserves products, so it also preserves graph products. We obtain

Proposition

For a flag complex \mathcal{K} , there is an isomorphism

$$\mathcal{U}(\text{gr}_\bullet \gamma(RA_{\mathcal{K}})) \cong H_*(\Omega(S^{2p+1})^{\mathcal{K}}; \mathbb{Z})$$

of associative \mathbb{Z} -algebras, where $p \geq 1$.

The lower central series of a right-angled Coxeter group $RC_{\mathcal{K}}$ is more subtle. One has $(\gamma_k(RC_{\mathcal{K}}))^2 \subset \gamma_{k+1}(RC_{\mathcal{K}})$, so $\text{gr}_\bullet \gamma(RC_{\mathcal{K}})$ is a Lie algebra over \mathbb{Z}_2 . There is an epimorphism

$$FL_{\mathbb{Z}_2}(v_1, \dots, v_m) / ([v_i, v_j] = 0 \text{ for } \{i, j\} \in \mathcal{K}) \rightarrow \text{gr}_\bullet \gamma(RC_{\mathcal{K}})$$

of Lie algebras over \mathbb{Z}_2 , where $FL_{\mathbb{Z}_2}(v_1, \dots, v_m) = FL(v_1, \dots, v_m) \otimes \mathbb{Z}_2$. This fails to be an isomorphism already when \mathcal{K} is two disjoint points and $RC_{\mathcal{K}} = \mathbb{Z}_2 \star \mathbb{Z}_2$ [Veryovkin].

Restricted Lie algebras and p -central series

Let p be a prime and let R be a field of characteristic p .

Restricted Lie algebras and p -central series

Let p be a prime and let R be a field of characteristic p .

A **restricted Lie algebra** over R , or shortly a **p -Lie algebra**, is a Lie algebra L over R equipped with a **p -power** operation $x \mapsto x^{[p]}$ satisfying

Restricted Lie algebras and p -central series

Let p be a prime and let R be a field of characteristic p .

A **restricted Lie algebra** over R , or shortly a **p -Lie algebra**, is a Lie algebra L over R equipped with a **p -power** operation $x \mapsto x^{[p]}$ satisfying

① $[x, y^{[p]}] = [\dots \underbrace{[[x, y], y], \dots, y}]_p$ for $x, y \in L$;

② $(\alpha x)^{[p]} = \alpha^p x^{[p]}$ for $x \in L$ and $\alpha \in R$;

③ $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y),$

where $s_i(x, y)$ is defined by the expansion

$$[\dots \underbrace{[[x, \lambda x + y], \lambda x + y], \dots, \lambda x + y}]_{p-1} = \sum_{i=1}^{p-1} i s_i(x, y) \lambda^{i-1}$$

for $x, y \in L$ and $\lambda \in R$.

Restricted Lie algebras and p -central series

Let p be a prime and let R be a field of characteristic p .

A **restricted Lie algebra** over R , or shortly a **p -Lie algebra**, is a Lie algebra L over R equipped with a **p -power** operation $x \mapsto x^{[p]}$ satisfying

① $[x, y^{[p]}] = [\dots \underbrace{[[x, y], y], \dots, y}]_p$ for $x, y \in L$;

② $(\alpha x)^{[p]} = \alpha^p x^{[p]}$ for $x \in L$ and $\alpha \in R$;

③ $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$,

where $s_i(x, y)$ is defined by the expansion

$$[\dots \underbrace{[[x, \lambda x + y], \lambda x + y], \dots, \lambda x + y}]_{p-1} = \sum_{i=1}^{p-1} i s_i(x, y) \lambda^{i-1}$$

for $x, y \in L$ and $\lambda \in R$.

For example, if $p = 2$ then $(x + y)^{[2]} = x^{[2]} + y^{[2]} + [x, y]$, and if $p = 3$ then $(x + y)^{[3]} = x^{[3]} + y^{[3]} + [[x, y], y] - [[x, y], x]$.

A **p -central series** on G is a central series $\Gamma^{[p]}(G) = \{\Gamma_k^{[p]}(G) : k \geq 1\}$ such that if $g \in \Gamma_k^{[p]}(G)$, then $g^p \in \Gamma_{pk}^{[p]}(G)$ for all k .

A **p -central series** on G is a central series $\Gamma^{[p]}(G) = \{\Gamma_k^{[p]}(G) : k \geq 1\}$ such that if $g \in \Gamma_k^{[p]}(G)$, then $g^p \in \Gamma_{pk}^{[p]}(G)$ for all k .

Each quotient $\Gamma_k^{[p]}(G)/\Gamma_{k+1}^{[p]}(G)$ is a \mathbb{Z}_p -module.

A **p -central series** on G is a central series $\Gamma^{[p]}(G) = \{\Gamma_k^{[p]}(G) : k \geq 1\}$ such that if $g \in \Gamma_k^{[p]}(G)$, then $g^p \in \Gamma_{pk}^{[p]}(G)$ for all k .

Each quotient $\Gamma_k^{[p]}(G)/\Gamma_{k+1}^{[p]}(G)$ is a \mathbb{Z}_p -module.

The **lower p -central series** is the fastest descending p -central series. It is given by $\gamma_1^{[p]}(G) = G$ and $\gamma_k^{[p]}(G) = (\gamma_{k-1}^{[p]}(G), G)(\gamma_{\lceil k/p \rceil}^{[p]})^p$.

A **p -central series** on G is a central series $\Gamma^{[p]}(G) = \{\Gamma_k^{[p]}(G) : k \geq 1\}$ such that if $g \in \Gamma_k^{[p]}(G)$, then $g^p \in \Gamma_{pk}^{[p]}(G)$ for all k .

Each quotient $\Gamma_k^{[p]}(G)/\Gamma_{k+1}^{[p]}(G)$ is a \mathbb{Z}_p -module.

The **lower p -central series** is the fastest descending p -central series. It is given by $\gamma_1^{[p]}(G) = G$ and $\gamma_k^{[p]}(G) = (\gamma_{k-1}^{[p]}(G), G)(\gamma_{\lceil k/p \rceil}^{[p]})^p$. Explicitly,

$$\gamma_k^{[p]}(G) = \prod_{mp^i \geq k} (\gamma_m(G))^{p^i}.$$

A **p -central series** on G is a central series $\Gamma^{[p]}(G) = \{\Gamma_k^{[p]}(G) : k \geq 1\}$ such that if $g \in \Gamma_k^{[p]}(G)$, then $g^p \in \Gamma_{pk}^{[p]}(G)$ for all k .

Each quotient $\Gamma_k^{[p]}(G)/\Gamma_{k+1}^{[p]}(G)$ is a \mathbb{Z}_p -module.

The **lower p -central series** is the fastest descending p -central series. It is given by $\gamma_1^{[p]}(G) = G$ and $\gamma_k^{[p]}(G) = (\gamma_{k-1}^{[p]}(G), G)(\gamma_{[k/p]}^{[p]})^p$. Explicitly,

$$\gamma_k^{[p]}(G) = \prod_{mp^i \geq k} (\gamma_m(G))^{p^i}.$$

The **associated p -Lie algebra** is the graded \mathbb{Z}_p -module

$$\text{gr} \bullet \Gamma^{[p]}(G) = \bigoplus_{k \geq 1} \Gamma_k^{[p]}(G)/\Gamma_{k+1}^{[p]}(G)$$

A **p -central series** on G is a central series $\Gamma^{[p]}(G) = \{\Gamma_k^{[p]}(G) : k \geq 1\}$ such that if $g \in \Gamma_k^{[p]}(G)$, then $g^p \in \Gamma_{pk}^{[p]}(G)$ for all k .

Each quotient $\Gamma_k^{[p]}(G)/\Gamma_{k+1}^{[p]}(G)$ is a \mathbb{Z}_p -module.

The **lower p -central series** is the fastest descending p -central series. It is given by $\gamma_1^{[p]}(G) = G$ and $\gamma_k^{[p]}(G) = (\gamma_{k-1}^{[p]}(G), G)(\gamma_{[k/p]}^{[p]})^p$. Explicitly,

$$\gamma_k^{[p]}(G) = \prod_{mp^i \geq k} (\gamma_m(G))^{p^i}.$$

The **associated p -Lie algebra** is the graded \mathbb{Z}_p -module

$$\text{gr} \bullet \Gamma^{[p]}(G) = \bigoplus_{k \geq 1} \Gamma_k^{[p]}(G)/\Gamma_{k+1}^{[p]}(G)$$

with the bracket and p -power defined by

$$[\bar{g}_k, \bar{g}_\ell] = \overline{(g_k, g_\ell)}, \quad (\bar{g}_k)^{[p]} = \overline{g_k^p}.$$

Recall $RC_{\mathcal{K}} = \mathbb{Z}_2^{\mathcal{K}}$. More generally, consider

$$\mathbb{Z}_{\rho}^{\mathcal{K}} = \operatorname{colim}_{I \in \mathcal{K}}^{\text{GRP}} \mathbb{Z}_{\rho}^I = F(a_1, \dots, a_m) / (a_i^{\rho} = 1, a_i a_j = a_j a_i \text{ for } \{i, j\} \in \mathcal{K}).$$

Recall $RC_{\mathcal{K}} = \mathbb{Z}_2^{\mathcal{K}}$. More generally, consider

$$\mathbb{Z}_p^{\mathcal{K}} = \operatorname{colim}_{I \in \mathcal{K}}^{\text{GRP}} \mathbb{Z}_p^I = F(a_1, \dots, a_m) / (a_i^p = 1, a_i a_j = a_j a_i \text{ for } \{i, j\} \in \mathcal{K}).$$

Theorem (P-Rahmatullaev)

For any simplicial complex \mathcal{K} and prime p , there is an isomorphism

$$\operatorname{gr}_{\bullet} \gamma^{[p]}(\mathbb{Z}_p^{\mathcal{K}}) \cong FL_p(u_1, \dots, u_m) / (u_i^{[p]} = 0, [u_i, u_j] = 0 \text{ for } \{i, j\} \in \mathcal{K})$$

of p -Lie algebras over \mathbb{Z}_p .

Recall $RC_{\mathcal{K}} = \mathbb{Z}_2^{\mathcal{K}}$. More generally, consider

$$\mathbb{Z}_p^{\mathcal{K}} = \operatorname{colim}_{I \in \mathcal{K}}^{\text{GRP}} \mathbb{Z}_p^I = F(a_1, \dots, a_m) / (a_i^p = 1, a_i a_j = a_j a_i \text{ for } \{i, j\} \in \mathcal{K}).$$

Theorem (P-Rahmatullaev)

For any simplicial complex \mathcal{K} and prime p , there is an isomorphism

$$\operatorname{gr}_{\bullet} \gamma^{[p]}(\mathbb{Z}_p^{\mathcal{K}}) \cong FL_p(u_1, \dots, u_m) / (u_i^{[p]} = 0, [u_i, u_j] = 0 \text{ for } \{i, j\} \in \mathcal{K})$$

of p -Lie algebras over \mathbb{Z}_p .

The proof uses Quillen's isomorphism of graded R -algebras

$$\mathcal{U}_p(\operatorname{gr}_{\bullet} \gamma^{[p]}(G)) \otimes_{\mathbb{Z}} R \cong \operatorname{gr}_{\bullet}(R[G]),$$

where $\mathcal{U}_p(-)$ is the universal enveloping of a p -Lie algebra and $\operatorname{gr}_{\bullet}(R[G])$ is the graded ring associated with the filtration of the group algebra $R[G]$ by the powers of the augmentation ideal.

Proposition

The functor $\text{gr}_\bullet \gamma^{[p]}: \text{GRP} \rightarrow \text{LIE}_p$ preserves graph products of elementary abelian p -groups.

Proposition

The functor $\text{gr}_\bullet \gamma^{[p]}: \text{GRP} \rightarrow \text{LIE}_p$ preserves graph products of elementary abelian p -groups.

Theorem

For any simplicial complex \mathcal{K} , there is an isomorphism

$$\text{gr}_\bullet \gamma^{[2]}(RC_{\mathcal{K}}) \cong FL_2(u_1, \dots, u_m) / (u_i^{[2]} = 0, [u_i, u_j] = 0 \text{ for } \{i, j\} \in \mathcal{K})$$

of 2-Lie algebras over \mathbb{Z}_2 .

Proposition

The functor $\text{gr}_\bullet \gamma^{[p]}: \text{GRP} \rightarrow \text{LIE}_p$ preserves graph products of elementary abelian p -groups.

Theorem

For any simplicial complex \mathcal{K} , there is an isomorphism

$$\text{gr}_\bullet \gamma^{[2]}(RC_{\mathcal{K}}) \cong FL_2(u_1, \dots, u_m) / (u_i^{[2]} = 0, [u_i, u_j] = 0 \text{ for } \{i, j\} \in \mathcal{K})$$

of 2-Lie algebras over \mathbb{Z}_2 .

Proposition

For a flag complex \mathcal{K} , there is an isomorphism

$$\mathcal{U}_2(\text{gr}_\bullet \gamma^{[2]}(RC_{\mathcal{K}})) \cong H_*(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}}; \mathbb{Z}_2)$$

of associative \mathbb{Z}_2 -algebras.

References

- [1] T. Panov, N. Ray and R. Vogt. *Colimits, Stanley-Reiner algebras, and loop spaces*, in: “Categorical Decomposition Techniques in Algebraic Topology” (G. Arone et al eds.), Progress in Mathematics, vol. 215, Birkhauser, Basel, 2004, pp. 261–291.
- [2] J. Grbić, T. Panov, S. Theriault and J. Wu. *The homotopy types of moment-angle complexes for flag complexes*. Trans. of the Amer. Math. Soc. 368 (2016), no. 9, 6663–6682.
- [3] T. Panov and Ya. Veryovkin. [On the commutator subgroup of a right-angled Artin group. J. Algebra 521 (2019), 284–298.
- [4] T. Panov and T. Rahmatullaev. *Polyhedral products, graph products and p -central series*. Proceedings of the Steklov Institute of Mathematics 326 (2024), 269–285.