On state-sum representations of quantum link invariants

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State-sum model for Link invariants

- **Definition:**
  A state sum model for a given link invariant is a sum over evaluations of combinatorial configurations (called states) related to the given link diagram, such that this sum is equal to the invariant that we wish to compute.

- **Example:** A state -sum model for Jones polynomial is given by Kauffman states. Using Tait graph, one could represent Kauffman states in terms of states coming from Tait graph.

Some definitions on directed graphs

Let $G = (V, E)$ be a directed graph.

- A map $f : E \to \mathbb{Z}$ is called an *edge coloring*.
- An edge coloring is called a *flow* if the image of f is in $\{0, 1, 2, , \ldots\}$, and for each vertex v

$$\sum_{e \text{ ingoing } v} f(e) = \sum_{e \text{ outgoing } v} f(e) =: d(v)$$

- An n-flow is a flow with $0 \leq d(v) \leq n$ for all vertices v.
- The set of flows respectively n-flows on a directed graph G is denoted by $\mathcal{F}(G)$ respectively $\mathcal{F}_n(G)$.
- Let $\mathcal{F}^n(G)$ denote the set of flows f on G with $f(e) \leq n$ for all e, the n-bounded flows. Note: $\mathcal{F}(G) \supset \mathcal{F}_n(G) \supset \mathcal{F}^n(G)$.
- The set $\mathcal{F}_1(G)$ is in bijective correspondence with the set of unions of vertex-disjoint directed cycles of G.

State-sum model for an invariant corresponding to a directed graph

Let $G = (V, E)$ be a directed graph. Let $\mathcal{I}$ be a quantum invariant defined from the d dimensional irreducible representation of the lie algebra $\mathfrak{g}$. Let $R = \mathbb{Z}[q^{\pm\frac{1}{d}}])$.

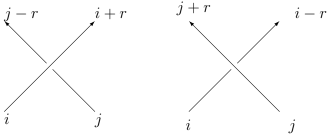
- **Definition:** A *state-sum model* for $\mathcal{I}$ in R on the directed graph $G = (V, E)$ assigns the state-sum

$$Z(G) = \sum_{f \in \mathrm{St}(G)} \Delta(f)\beta(f) = \sum_{f \in \tilde{\mathrm{St}}(G)} \Delta(f)\beta(f) \in R.$$

- Here the set of all states $\mathrm{St}(G) \subset \{f : E \to \mathbb{Z}\}$ is a set of *edge-colorings*, $\beta(f) := \prod_{v \in V} \beta_v(f)$, with $\beta_v(f) \in R$ defined *locally*, the factor $\Delta(f) \neq 0$ is defined from edge colorings and *geometry* and/or extended structure on the graph.
- The set of contributory states $\tilde{\mathrm{St}}(G) = \{f \in \mathrm{St} | \beta(f) \neq 0\} \subset \mathrm{St}(G)$.

The Part-arc graph $P\mathcal{K}$

- The part-arc graph $P\mathcal{K}$ is a plane 4-regular directed graph with the sign at each vertex giving the projection $\mathcal{K}$.
- We let $\mathcal{F}^n_{\pm}(P\mathcal{K})$ denote the set of all n-bounded flows on $P\mathcal{K}$, which are nondecreasing/nonincreasing along the overcrossing at each vertex. We denote these flows correspondingly $+/-$. Thus $f \in \mathcal{F}^n_{+}(P\mathcal{K})$ looks as in Figure 1 (a) (for positive) and Figure 1 (b) for negative crossings with $r \geq 0$ (and all edge labels ≥ 0 because we have a flow).



$Figure 1(a)$ $Figure 1(b)$

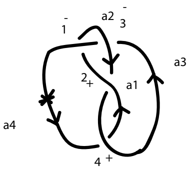
The Signed-arc graph $G\mathcal{K}$

For a knot diagram $\mathcal{K}$, $G_{\mathcal{K}}$ is defined as a directed bichromatic *labeled* fat-graph with:

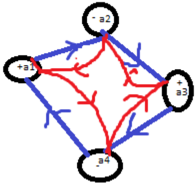
- fat vertices v correspond to overcrossing respectively undercrossing arcs with signs corresponding to the sign of the crossing at which an overcrossing arc ends, the boundary of each fat vertex oriented
- blue edges corresponding to the transition between overcrossing/undercrossing arcs, so for each vertex v of $G\mathcal{K}$ there is a unique outgoing blue edge e_v^b,and a unique incoming blue edge
- a unique outgoing red edge e_v^r for the transition of jumping up to the overcrossing arc respectively jumping down to the undercrossing arc at the end of the arc of $P\mathcal{K}$ corresponding to v
- an ordering of the incoming edges along the fat vertex
- *rotation* numbers assigned to all edges in the boundary of fat vertices, between the blue edges following the orientation

$G\mathcal{K}$ for Figure eight knot

$\mathcal{K}$:



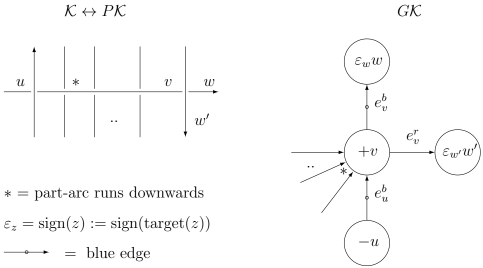
G_K :



Correspondence between $P\mathcal{K}$ and $G\mathcal{K}$

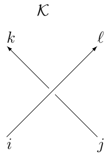
Lemma: *There is a natural bijective map from edge-colorings of $G\mathcal{K}$ to edge-colorings of $P\mathcal{K}$, which assigns the label of a blue edge of $G\mathcal{K}$ to be the label of the edge of $P\mathcal{K}$ following the target of the edge, and to each red edge label assigns the change in the labels of the two part-arcs with respect to the standard orientation along fat vertices. This bijection maps*

$$\mathcal{F}_n(G\mathcal{K}) \to \mathcal{F}^n(P\mathcal{K})$$



Flows on $G\mathcal{K}$

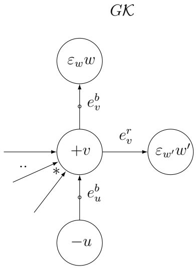
$P\mathcal{K}$ has the distinguished part-arc $*_0$, which is the rightmost arc running downwards. We will consider n-bounded flows on $P\mathcal{K}$, which take the value 0 on $*_0$. By the flow lemma these correspond to the set $\mathcal{F}_n^*(G\mathcal{K}) \subset \mathcal{F}_n(G\mathcal{K})$ defined by zero flow on blue and red edges *preceding* $*_0$ along the fat vertex corresponding to the over crossing arc containing $*_0$.



flow at u:

$j = j_0 + \Delta_1 + \ldots + \Delta_r = \ell - i + k$

$\Longrightarrow j - k = \ell - i \geq 0$ ((+)-flow)



Some notations

- For an r-string braid with standard braid projection $\mathcal{K}$ we define $\delta(\mathcal{K}) := \frac{1}{2}(r - 1 + \omega(\mathcal{K}))$, where $\omega(\mathcal{K})$ is the writhe of the projection, which is of course also determined by b.
- For each flow $f \in \mathcal{F}_n(G\mathcal{K})$ let

$$\mathrm{rot}(f) := \sum_e f(e)\mathrm{rot}(e), \qquad \mathrm{exc}(f) := \sum_v \mathrm{sign}(v) f(e_v^b) \sum_{e<e_v^r} f(e)$$

  Let $\delta(f) := -(rot(f) + exc(f))$.
- For $k \in \mathbb{Z}$ let $(k)_t := \frac{t^k-1}{t-1}$ and $(k)_t! := (1)_t(2)_t \cdots (k)_t, \quad \binom{k}{\ell}_t := \frac{(k)_t!}{(\ell)_t!(k-\ell)_t!}$.
- Define for $n \geq 1$ and $v \in G\mathcal{K}$, $\beta_n(f,v) := \binom{d(v)}{f(e_v^b)}_{t^{-1}} t^{n\cdot\mathrm{sign}(v)f(e_v^b)} \prod_{k=0}^{f(e_v^r)-1} \left(1 - t^{\mathrm{sign}(v)(n-\sum_{e<e_v^r} f(e)-k)}\right)$ and $\beta_n(f) := \prod_v \beta_n(f,v)$.

Application to the colored Jones polynomial

Given a braid projection $\mathcal{K}$ of a link K and $\delta(\mathcal{K}), \delta(f)$ as before, for $n \geq 1$ let J_n be the unframed normalized n-colored Jones polynomial with $J = J_1$ the classical Jones polynomial (with skein relation $t^{-1}J(K_+) - tJ(K_-) = (t^{1/2} - t^{-1/2})J(K_0)$, $J_n(U) = 1$ for unknot U and $n \geq 1$) is given by

$$J_n(K) = (-1)^{2\delta(\mathcal{K})} t^{n\delta(\mathcal{K})} \sum_{f \in \mathcal{F}_n^*(G\mathcal{K})} t^{\delta(f)} \beta_n(f) \in \mathbb{Z}[t^{\pm 1/2}]$$

Computing $J_n(W(3,m))$

- For $\ell \geq 1$ let $K = (\sigma_1^{-1}\sigma_2)^{3\ell+1} = \overline{W(3,3\ell+1)}$ with corresponding projection $\mathcal{K}$. Note that $K = W(3,m)$ is amphichiral and therefore $J_n(K) = J_n(\overline{K})$.
- We have $\delta(\mathcal{K}) = 1$ and $\delta(\mathbf{j}) = -(i_{2\ell+1} + i_{4\ell+1} + j_{6\ell+1}) + \sum_{k=1}^{6\ell+1} (-1)^k i_k i_{\tau^{-1}(k)-1}$.
- $\mathbf{0} \leq \mathbf{k} \leq \mathbf{n}$ for a multi-index $\mathbf{k}$ is component-wise, $\mathbf{0}, \mathbf{n}$ are constant $0, n$-sequences. Then for $k = 1, \ldots, 6\ell+1$ we have $\beta_n(\mathbf{j}, k) =$
$\binom{i_k + j_k}{i_k}_{t^{-1}} t^{n(-1)^{k+1} i_k} \prod_{r=0}^{j_k - 1} \left(1 - t^{(-1)^{k+1}\left(n - i_{\tau^{-1}(k)-1} - r\right)}\right)$
- Substituting these values in the formula on the previous page we obtain a multi-sum formula for the colored Jones polynomial for $W(3,m)$.