Complexity theory of 3-manifolds and its tabulation

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A few words about the problem

- Denote by *M* the set of all compact 3-manifolds. We wish to study it systematically. The crucial question is the choice of filtration in *M* (decomposition at *M* separated pieces). A useful tool here would be a measure of “complexity” of a 3-manifold. Given such a measure, we might hope to enumerate all “simple” manifolds before moving on to more complicated ones.
- There are several well-known candidates for complexity: Heegard genus, number of simplieces in a triangulation of M.

Complexity of 3-manifolds

is a map from the set of all compact 3-manifolds to the set of non-negative integers

$$c: \{M\} \to N \cup \{0\}$$

Desirable properties of complexity:

Naturalness:

- c agrees with the complexity in a non-formal meaning of the expression;

- c is decreased under simplification moves.

Finiteness: the number of objects having a fixed complexity is finite.

Construction of a complexity function

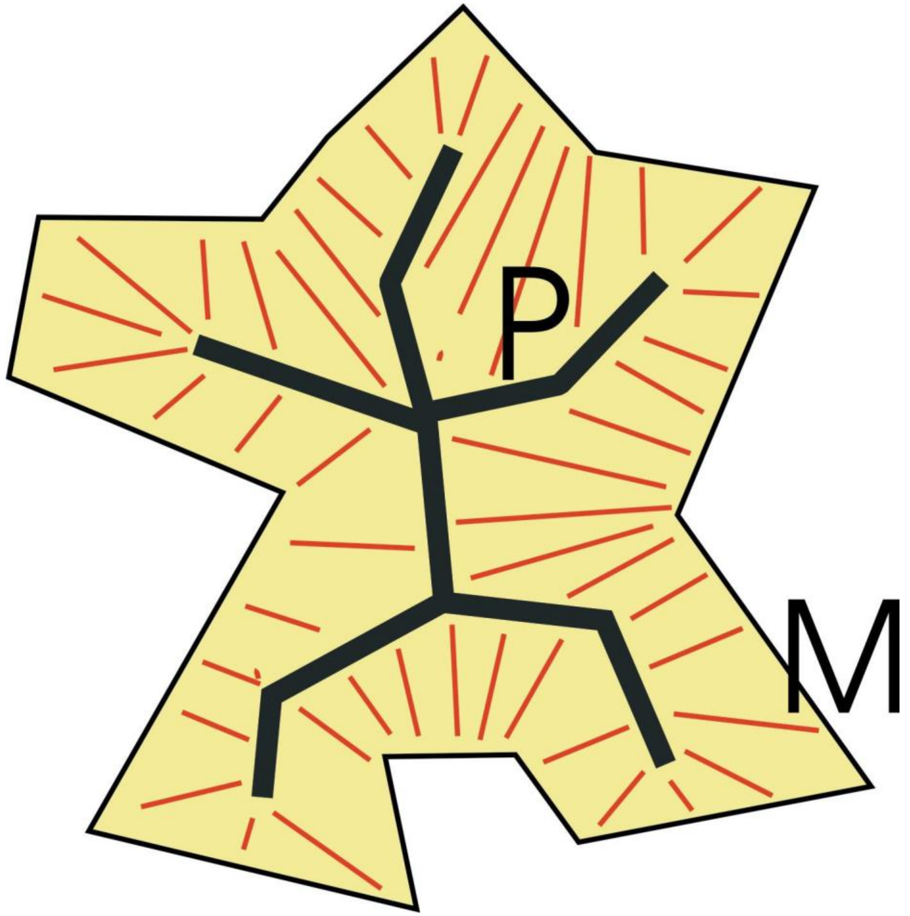
Idea: Let us count singularities of the given object X.
We get a non-negative number C and call it complexity of X

Construction of the complexity

- P is a spine of M, if M \ P=∂M x (0,1]

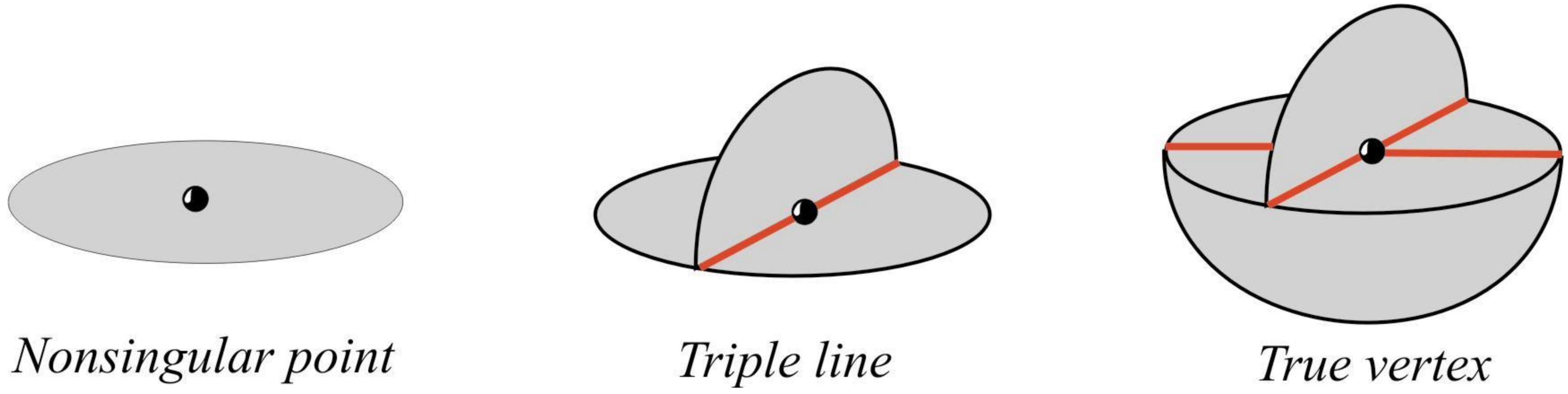
- See book:

S. V. Matveev, Algorithmic topology and classification of 3-manifolds. Algorithms and Computation in Mathematics, 9. Springer-Verlag, Berlin, (2003) xii+478 pp. ISBN: 3-540-44171-9



Special spine

- 1. It is simple, i.e. has a nice local structure



- 2. The set of nonsingular points is a union of open 2-cells
- 3. The set of true vertices is called *special graph.*

Definition

$$C(M)=\min V(P)$$

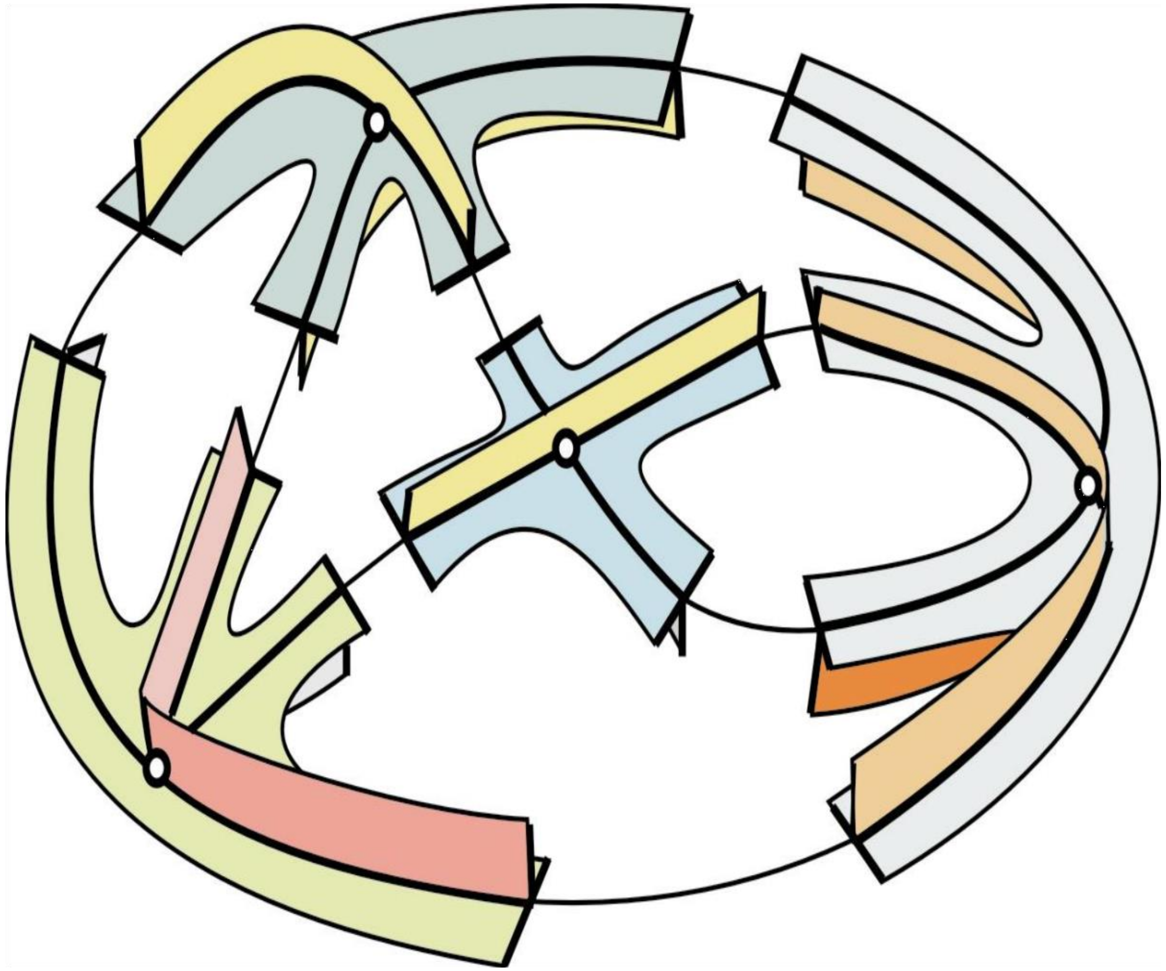
where V(P) is the number of true vertices of P and the minimum is taken over all simple spines of M

Construction of 3-manifolds

- First, we build a special graph with k true vertices.
- Then, on this graph, we construct neighborhoods of true vertices, connecting these vertices.
- After that we build a special spine.

Then we Remove duplicates.

Recognize the manifolds using computer



On the table of 3-manifolds of complexity $\leq$ 13

- The table shows the number of 3-manifolds depending on the complexity.
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c	0	1	2	3	4	5	6	7	8	9	10	11	12	13	Total
N(c)	3	2	4	7	14	31	74	175	436	1154	3078	8421	23448	66197	103041

Problem:

Why all tabulated 3-manifolds are different ?

Answer:

With a few exceptions, all of them have different quantum invariants.

We have used TV-invariants (they are parameterized by natural numbers)

An information about manifolds of complexity ≤ 13 from the point of view of Thurston's classification

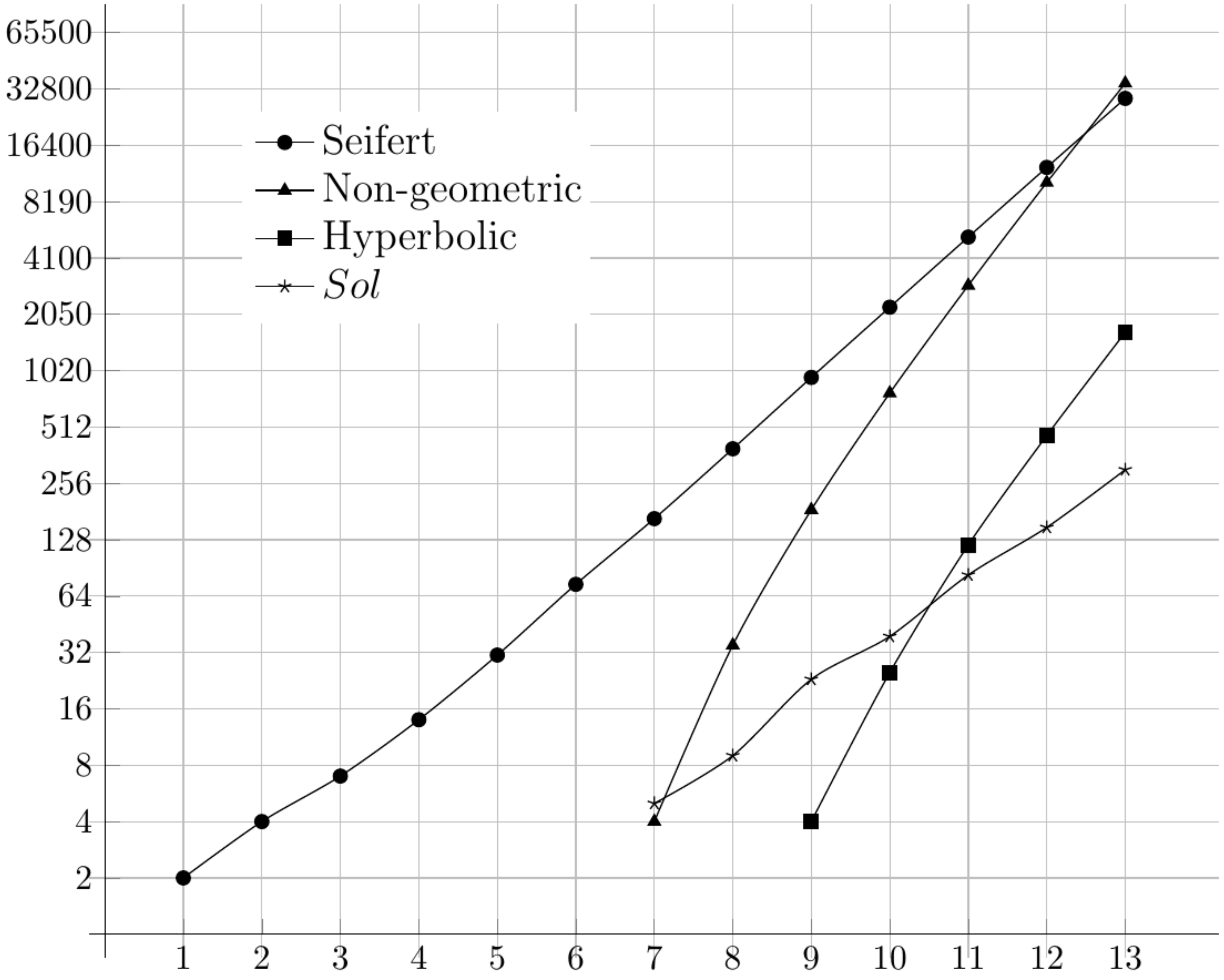
- Recall that Thurston proved that there are 8 geometries: $E^3,\ S^3$, $S^2 \times R,\ H^2 \times R,\ \widetilde{SL_2 R},\ Nil,\ Sol,\ and\ H^3$.
- (Thurston, W., *Three dimensional manifolds, Kleinian groups and*

hyperbolic geometry. Bull. Amer. Math. Soc. **6**(1982), 357--381.)

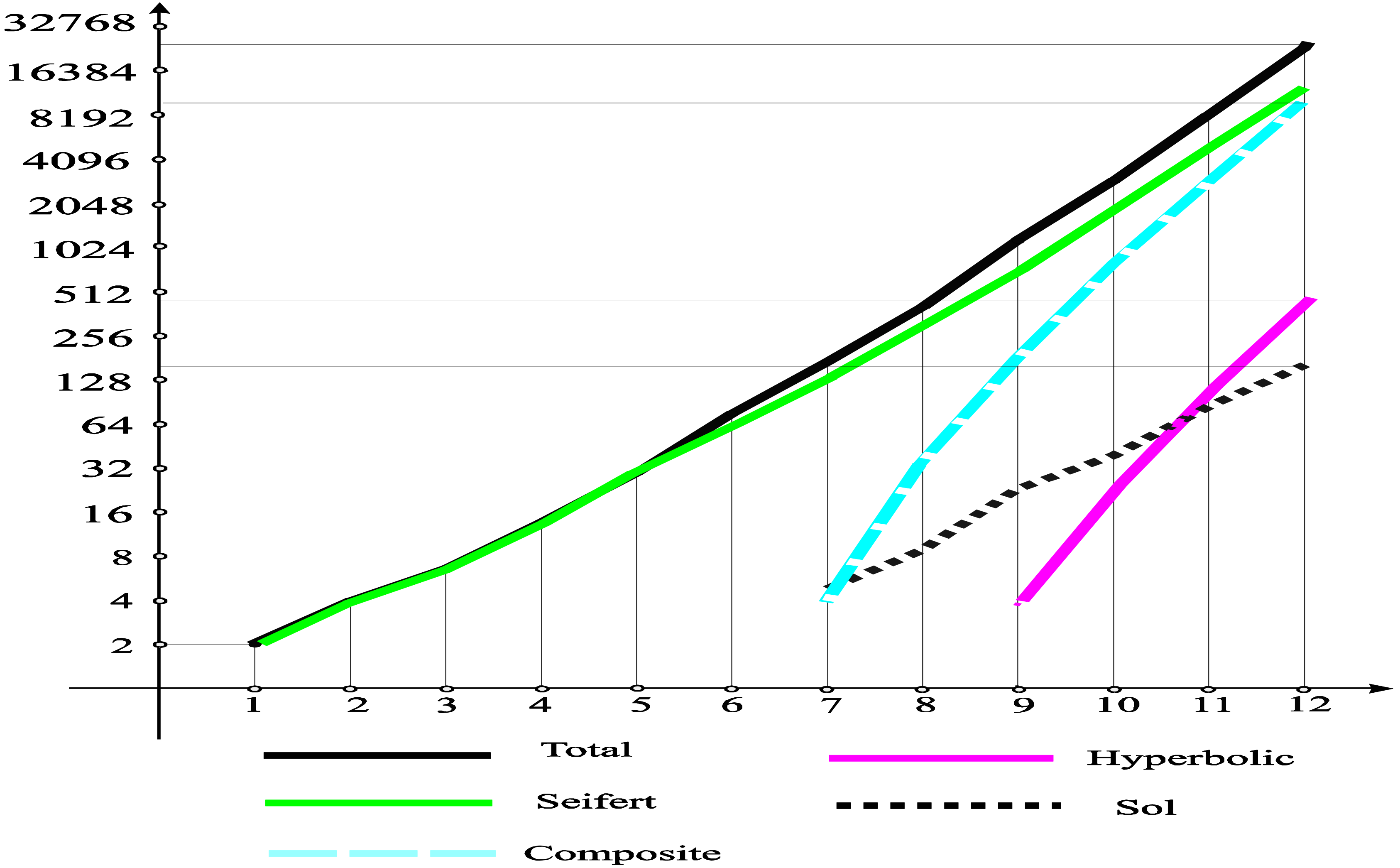
- A 3-manifold allows not more than one of them.
- Following table gives an information about manifolds of complexity $\leq$13 according to the classification of Thurston.

c	$S^2 \times R$	E^3	$H^2 \times R$	S^3	$\widetilde{SL_2R}$	Nil	Sol	H^3	Nongeo metric
0	0	0	0	3	0	0	0	0	0
1	0	0	0	2	0	0	0	0	0
2	0	0	0	4	0	0	0	0	0
3	0	0	0	7	0	0	0	0	0
4	0	0	0	14	0	0	0	0	0
5	0	0	0	31	0	0	0	0	0
6	0	0	0	61	0	7	0	0	0
7	0	0	0	117	39	10	5	0	4
8	0	0	2	214	162	14	9	0	35
9	0	0	0	414	513	15	23	4	185
10	0	0	8	798	1416	15	39	25	777
11	0	0	4	1582	3696	15	83	120	2921
12	0	0	24	3118	9324	15	149	461	10357
13	0	0	9	6222	22916	15	303	1641	35091

An example of the distribution of manifolds up to complexity 13 is shown in the figure. The graphs show how the number of manifolds grows with increasing complexity.



An example of the distribution of manifolds up to complexity 12 is shown in the figure. The graphs show how the number of varieties grows with increasing complexity.

- 

- The result related to complexity ≤ 12 is published, for example, in
- Vesnin, A. Yu.; Matveev, S. V.; Fominykh, E. A., *New aspects of*
- *the complexity theory of three-dimensional manifolds.* (Russian) Uspekhi Mat. Nauk **73** (2018), no. 4(442), 53--102; translation in Russian Math. Surveys **73** (2018), no. 4, 615--660

The result related to complexity ≤ 13 is published, for example, S.V. Matveev, V.V. Tarkaev, 2020, *Recognition and tabulation of 3-dimensional manifolds up to complexity 13.* Chebyshevskii sbornik, vol. 21, no.2, pp. 280-290.

- Thank you for your attention