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ON 3-STRAND SINGULAR PURE BRAID GROUP

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Joint work with Valeriy Bardakov

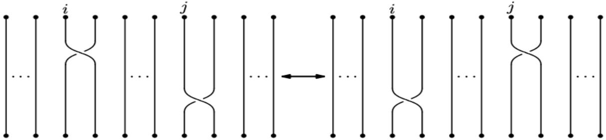
Artin braid groups

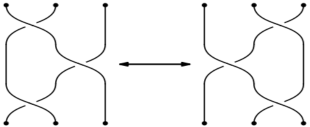
Artin(1925)

The braid group B_n, $n \geq 2$, on n strings can be defined as a group generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ with the defining relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j| \geq 2.$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad i = 1, 2, \ldots, n-2,$$

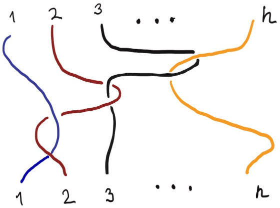




Braid group relations

PURE BRAID GROUP

There is a homomorphism $\varphi : B_n \to S_n$, $\varphi(\sigma_i) = (i, i+1)$, $i = 1, 2, \ldots, n-1$. Its kernel $Ker(\varphi)$ is called the *pure braid group* and denoted by P_n.



The group P_n is generated by a_{ij}, $1 \leq i < j \leq n$. These generators can be expressed by the generators of B_n as follows

$$a_{i,i+1} = \sigma_i^2,$$

$$a_{ij} = \sigma_{j-1}\,\sigma_{j-2}\ldots\sigma_{i+1}\,\sigma_i^2\,\sigma_{i+1}^{-1}\ldots\sigma_{j-2}^{-1}\,\sigma_{j-1}^{-1}, \quad i+1 < j \leq n.$$

PURE BRAID GROUP

The pure braid group P_n is defined by the relations (for $\varepsilon = \pm 1$)

$$a_{ik}^{-\varepsilon} a_{kj} a_{ik}^{\varepsilon} = (a_{ij} a_{kj})^{\varepsilon} a_{kj} (a_{ij} a_{kj})^{-\varepsilon},$$

$$a_{km}^{-\varepsilon} a_{kj} a_{km}^{\varepsilon} = (a_{kj} a_{mj})^{\varepsilon} a_{kj} (a_{kj} a_{mj})^{-\varepsilon}, \text{ при } m < j,$$

$$a_{im}^{-\varepsilon} a_{kj} a_{im}^{\varepsilon} = [a_{ij}^{-\varepsilon}, a_{mj}^{-\varepsilon}]^{\varepsilon} a_{kj} [a_{ij}^{-\varepsilon}, a_{mj}^{-\varepsilon}]^{-\varepsilon}, \text{ при } i < k < m,$$

$$a_{im}^{-\varepsilon} a_{kj} a_{im}^{\varepsilon} = a_{kj}, \text{ при } k < i < m < j \text{ или } m < k.$$

SINGULAR BRAID GROUP

J. C. Baez(1992), J. S. Birman(1993)

The Baez–Birman monoid or the singular braid monoid SB_n is generated (as a monoid) by elements σ_i, σ_i^{-1}, τ_i, $i = 1, 2, \ldots, n-1$ and the defining relations:

$$\tau_i \tau_j = \tau_j \tau_i, \quad |i-j| \geq 2,$$

$$\tau_i \sigma_j = \sigma_j \tau_i, \quad |i-j| \geq 2,$$

$$\tau_i \sigma_i = \sigma_i \tau_i, \quad i = 1, 2, \ldots, n-1,$$

$$\sigma_i \sigma_{i+1} \tau_i = \tau_{i+1} \sigma_i \sigma_{i+1}, \quad i = 1, 2, \ldots, n-2,$$

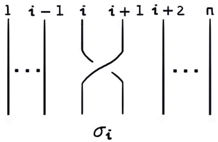
$$\sigma_{i+1} \sigma_i \tau_{i+1} = \tau_i \sigma_{i+1} \sigma_i, \quad i = 1, 2, \ldots, n-2.$$

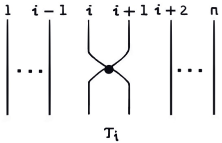
SINGULAR BRAID GROUP

R. Fenn, E. Keyman, C. Rourke (1997) proved that the singular braid monoid SB_n is embedded into the group SG_n which is called the singular braid group.

$$SB_n \to SG_n$$

Geometric interpretation





The elementary braids σ_i and τ_i

SINGULAR PURE BRAID GROUP

Define the map $\pi : SG_n \longrightarrow S_n$ of SG_n onto the symmetric group S_n on n symbols by actions the on generators

$$\pi(\sigma_i) = \pi(\tau_i) = (i, i+1), \quad i = 1, 2, \dots, n-1.$$

The kernel $\ker(\pi)$ of this map is called the *singular pure braid group* and denoted by SP_n.

SP_n is a normal subgroup of index $n!$ of SG_n.

REIDEMEISTER-SCHREIER METHOD

The set

$$\Lambda_n = \left\{ \prod_{k=2}^{n} m_{k,j_k} | 1 \leq j_k \leq k \right\}, m_{kl} = \rho_{k-1}\,\rho_{k-2}\ldots\rho_l, l < k; m_{kk} = 1$$

is a Schreier set of coset representatives of SP_n in SG_n.
$\eta : SG_n \longrightarrow \Lambda_n$, $w \in SG_n$, $\overline{w} \in \Lambda_n$. The element $w\overline{w}^{-1}$ belongs to SP_n.

$$S_{\lambda,a} = \lambda a \cdot (\overline{\lambda a})^{-1}, where$$

λ runs over the set Λ_n, a runs over the set of generators of SG_n.

REWRITING PROCESS

Let us associate to the reduced word

$$u = a_1^{\varepsilon_1}\, a_2^{\varepsilon_2} \ldots a_\nu^{\varepsilon_\nu}, \quad \varepsilon_l = \pm 1, \quad a_l \in \{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \tau_1, \tau_2, \ldots, \tau_{n-1}\},$$

the word

$$\tau(u) = S_{k_1,a_1}^{\varepsilon_1}\, S_{k_2,a_2}^{\varepsilon_2} \ldots S_{k_\nu,a_\nu}^{\varepsilon_\nu}$$

in the generators of SP_n, where k_j is a representative of the $(j-1)$th initial segment of the word u if $\varepsilon_j = 1$ and k_j is a representative of the jth initial segment of the word u if $\varepsilon_j = -1$.
The group SP_n is defined by relations

$$r_{\mu,\lambda} = \tau(\lambda\, r_\mu\, \lambda^{-1}), \quad \lambda \in \Lambda_n,$$

where r_μ is the defining relation of SG_n.

CASE $n = 2$.

$$SG_2 = \langle \sigma_1, \tau_1 \mid \sigma_1\tau_1 = \tau_1\sigma_1 \rangle \cong \mathbb{Z} \times \mathbb{Z}.$$

The set of coset representatives:

$$\Lambda_2 = \{1, \sigma_1\}.$$

$$S_{\lambda,a} = \lambda a \cdot (\overline{\lambda a})^{-1}, \quad \lambda \in \Lambda_2, \quad a \in \{\sigma_1, \tau_1\}.$$

$$S_{1,\sigma_1} = \sigma_1 \cdot (\overline{\sigma_1})^{-1} = \sigma_1 \cdot \sigma_1^{-1} = 1,$$

$$S_{1,\tau_1} = \tau_1 \cdot (\overline{\tau_1})^{-1} = \tau_1 \cdot \sigma_1^{-1},$$

$$S_{\sigma_1,\sigma_1} = \sigma_1^2 \cdot \overline{\sigma_1^2}^{-1} = \sigma_1^2 \cdot 1 = \sigma_1^2,$$

$$S_{\sigma_1,\tau_1} = \sigma_1\tau_1 \cdot (\overline{\sigma_1\tau_1})^{-1} = \sigma_1\tau_1.$$

SP_2 is generated by three elements:

$$S_{1,\tau_1} = \tau_1\sigma_1^{-1}, \quad S_{\sigma_1,\sigma_1} = \sigma_1^2, \quad S_{\sigma_1,\tau_1} = \sigma_1\tau_1.$$

The element $a_{12} = \sigma_1^2$ is a generator of the pure braid group P_2.

CASE $n = 2$.

$r = \sigma_1 \tau_1 \sigma_1^{-1} \tau_1^{-1}$

$$r = S_{1,\sigma_1} S_{\sigma_1,\tau_1} S_{\sigma_1,\sigma_1}^{-1} S_{1,\tau_1}^{-1} = S_{\sigma_1,\tau_1} S_{\sigma_1,\sigma_1}^{-1} S_{1,\tau_1}^{-1} = 1.$$

$$S_{1,\tau_1} = S_{\sigma_1,\tau_1} S_{\sigma_1,\sigma_1}^{-1}.$$

$$\sigma_1 r \sigma_1^{-1} = S_{1,\sigma_1} S_{\sigma_1,\sigma_1} S_{1,\tau_1} S_{1,\sigma_1}^{-1} S_{\sigma_1,\tau_1}^{-1} S_{1,\sigma_1}^{-1} = S_{\sigma_1,\sigma_1} S_{1,\tau_1} S_{\sigma_1,\tau_1}^{-1} = 1.$$

$$S_{\sigma_1,\sigma_1} S_{\sigma_1,\tau_1} = S_{\sigma_1,\tau_1} S_{\sigma_1,\sigma_1}.$$

Put $a_{12} = S_{\sigma_1,\sigma_1}$, $b_{12} = S_{\sigma_1,\tau_1}$.

CASE $n = 2$.

LEMMA. 1) $SP_2 = \langle a_{12}, b_{12} \mid a_{12}\, b_{12} = b_{12}\, a_{12} \rangle \cong \mathbb{Z} \times \mathbb{Z}$;
2) SP_2 is normal in SG_2 and the action of SG_2 on SP_2 is defined by the formulas

$$a_{12}^{\sigma_1} = a_{12}, \quad b_{12}^{\sigma_1} = b_{12},$$
$$a_{12}^{\tau_1} = a_{12}, \quad b_{12}^{\tau_1} = b_{12}.$$

CASE $n = 3$.

SG_3 is generated by elements

$$\sigma_1, \sigma_2, \tau_1, \tau_2,$$

and is defined by relations

$$\sigma_1\tau_1 = \tau_1\sigma_1, \quad \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \quad \sigma_2\tau_2 = \tau_2\sigma_2,$$

$$\sigma_1\sigma_2\tau_1 = \tau_2\sigma_1\sigma_2, \quad \sigma_2\sigma_1\tau_2 = \tau_1\sigma_2\sigma_1.$$

$$\Lambda_3 = \{1, \sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1\}.$$

The group SP_3 is generated by elements

$$S_{\lambda,a} = \lambda a \cdot (\overline{\lambda a})^{-1}, \quad \lambda \in \Lambda_3, \quad a \in \{\sigma_1, \sigma_2, \tau_1, \tau_2\}.$$

CASE $n = 3$.

$$S_{1,\sigma_1} = \sigma_1 \cdot (\overline{\sigma_1})^{-1} = \sigma_1 \cdot \sigma_1^{-1} = 1,$$
$$S_{1,\sigma_2} = \sigma_2 \cdot (\overline{\sigma_2})^{-1} = \sigma_2 \cdot \sigma_2^{-1} = 1,$$
$$S_{1,\tau_1} = \tau_1 \cdot (\overline{\tau_1})^{-1} = \tau_1 \cdot \sigma_1^{-1},$$
$$S_{1,\tau_2} = \tau_2 \cdot (\overline{\tau_2})^{-1} = \tau_2 \cdot \sigma_2^{-1},$$
$$S_{\sigma_1,\sigma_1} = \sigma_1^2 \cdot \overline{\sigma_1^2}^{-1} = \sigma_1^2 \cdot 1 = \sigma_1^2,$$
$$S_{\sigma_1,\sigma_2} = \sigma_1\sigma_2 \cdot (\overline{\sigma_1\sigma_2})^{-1} = 1,$$
$$S_{\sigma_1,\tau_1} = \sigma_1\tau_1 \cdot (\overline{\sigma_1\tau_1})^{-1} = \sigma_1\tau_1,$$
$$S_{\sigma_1,\tau_2} = \sigma_1\tau_2 \cdot (\overline{\sigma_1\tau_2})^{-1} = \sigma_1\tau_2\sigma_2^{-1}\sigma_1^{-1},$$
$$S_{\sigma_2,\sigma_1} = \sigma_2\sigma_1 \cdot (\overline{\sigma_2\sigma_1})^{-1} = 1,$$
$$S_{\sigma_2,\sigma_2} = \sigma_2^2 \cdot \overline{\sigma_2^2}^{-1} = \sigma_2^2 \cdot 1 = \sigma_2^2,$$
$$S_{\sigma_2,\tau_1} = \sigma_2\tau_1 \cdot (\overline{\sigma_2\sigma_1})^{-1} = \sigma_2\tau_1\sigma_1^{-1}\sigma_2^{-1},$$

CASE $n = 3$.

$$S_{\sigma_2,\tau_2} = \sigma_2\tau_2,$$
$$S_{\sigma_1\sigma_2,\sigma_1} = \sigma_1\sigma_2\sigma_1 \cdot (\sigma_1\sigma_2\sigma_1)^{-1} = 1,$$
$$S_{\sigma_1\sigma_2,\sigma_2} = \sigma_1\sigma_2^2\sigma_1^{-1},$$
$$S_{\sigma_1\sigma_2,\tau_1} = \sigma_1\sigma_2\tau_1\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1},$$
$$S_{\sigma_1\sigma_2,\tau_2} = \sigma_1\sigma_2\tau_2\sigma_1^{-1},$$
$$S_{\sigma_2\sigma_1,\sigma_1} = \sigma_2\sigma_1^2\sigma_2^{-1},$$
$$S_{\sigma_2\sigma_1,\sigma_2} = \sigma_2\sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1},$$
$$S_{\sigma_2\sigma_1,\tau_1} = \sigma_2\sigma_1\tau_1\sigma_2^{-1},$$
$$S_{\sigma_2\sigma_1,\tau_2} = \sigma_2\sigma_1\tau_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1},$$
$$S_{\sigma_1\sigma_2\sigma_1,\sigma_1} = \sigma_1\sigma_2\sigma_1^2\sigma_2^{-1}\sigma_1^{-1},$$
$$S_{\sigma_1\sigma_2\sigma_1,\sigma_2} = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1},$$
$$S_{\sigma_1\sigma_2\sigma_1,\tau_1} = \sigma_1\sigma_2\sigma_1\tau_1\sigma_2^{-1}\sigma_1^{-1}, S_{\sigma_1\sigma_2\sigma_1,\tau_2} = \sigma_1\sigma_2\sigma_1\tau_2\sigma_1^{-1}\sigma_2^{-1}.$$

CASE $n = 3$.

LEMMA. From relation $r_1 = \sigma_1\tau_1\sigma_1^{-1}\tau_1^{-1}$ follows 6 relations, applying which we can remove generators:

$$S_{1,\tau_1} = S_{\sigma_1,\tau_1}S^{-1}_{\sigma_1,\sigma_1},\ S_{\sigma_2,\tau_1} = S_{\sigma_2\sigma_1,\tau_1}S^{-1}_{\sigma_2\sigma_1,\sigma_1},\ S_{\sigma_1\sigma_2,\tau_1} = S_{\sigma_1\sigma_2\sigma_1,\tau_1}S^{-1}_{\sigma_1\sigma_2\sigma_1,\sigma_1},$$

and we get 3 relations:

$$S_{\sigma_1,\sigma_1}S_{\sigma_1,\tau_1} = S_{\sigma_1,\tau_1}S_{\sigma_1,\sigma_1},$$

$$S_{\sigma_2\sigma_1,\tau_1}S_{\sigma_2\sigma_1,\sigma_1} = S_{\sigma_2\sigma_1,\sigma_1}S_{\sigma_2\sigma_1,\tau_1},$$

$$S_{\sigma_2,\sigma_2}S_{\sigma_1\sigma_2\sigma_1,\tau_1}S^{-1}_{\sigma_2,\sigma_2} = S_{\sigma_1\sigma_2\sigma_1,\tau_1}.$$

CASE $n = 3$.

LEMMA. From relation $r_2 = \sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}$ follows 6 relations, applying which we can remove 4 generators:

$$S_{\sigma_2\sigma_1,\sigma_2} = 1,\ S_{\sigma_1\sigma_2\sigma_1,\sigma_2} = S_{\sigma_1,\sigma_1},\ S_{\sigma_1\sigma_2\sigma_1,\sigma_1} = S_{\sigma_2,\sigma_2},\ S_{\sigma_1\sigma_2,\sigma_2} = S_{\sigma_1,\sigma_1}S_{\sigma_2\sigma_1,\sigma_1}$$

and we get 2 relations:

$$S_{\sigma_1,\sigma_1}S_{\sigma_2\sigma_1,\sigma_1}S_{\sigma_1,\sigma_1}^{-1} = S_{\sigma_2,\sigma_2}^{-1}S_{\sigma_2\sigma_1,\sigma_1}S_{\sigma_2,\sigma_2}$$

$$S_{\sigma_2,\sigma_2}S_{\sigma_1,\sigma_1}S_{\sigma_2\sigma_1,\sigma_1} = S_{\sigma_1,\sigma_1}S_{\sigma_2\sigma_1,\sigma_1}S_{\sigma_2,\sigma_2}.$$

COROLLARY. The generators

$$S_{\sigma_1,\sigma_1} = a_{12},\ S_{\sigma_2\sigma_1,\sigma_1} = a_{13},\ S_{\sigma_2,\sigma_2} = a_{23}$$

satisfy relations

$$a_{12}a_{13}a_{12}^{-1} = a_{23}^{-1}a_{13}a_{23},\ a_{12}a_{23}a_{12}^{-1} = a_{23}^{-1}a_{13}^{-1}a_{23}a_{13}a_{23}.$$

CASE $n = 3$.

LEMMA. From relation $r_3 = \sigma_2\tau_2\sigma_2^{-1}\tau_2^{-1}$ follows 6 relations, applying which we can remove 3 generators:

$$S_{1,\tau_2} = S_{\sigma_2,\tau_2}S^{-1}_{\sigma_2,\sigma_2}, \quad S_{\sigma_1,\tau_2} = S_{\sigma_1\sigma_2,\tau_2}S_{\sigma_1,\sigma_1}S^{-1}_{\sigma_2\sigma_1,\sigma_1}S^{-1}_{\sigma_1,\sigma_1},$$

$$S_{\sigma_2\sigma_1,\tau_2} = S_{\sigma_1\sigma_2\sigma_1,\tau_2}S^{-1}_{\sigma_1,\sigma_1},$$

$$S_{\sigma_2,\sigma_2}S_{\sigma_2,\tau_2} = S_{\sigma_2,\tau_2}S_{\sigma_2,\sigma_2}.$$

$$S_{\sigma_1,\sigma_1}S_{\sigma_2\sigma_1,\sigma_1}S^{-1}_{\sigma_1,\sigma_1} \cdot S_{\sigma_1\sigma_2,\tau_2} \cdot S_{\sigma_1,\sigma_1}S^{-1}_{\sigma_2\sigma_1,\sigma_1}S^{-1}_{\sigma_1,\sigma_1} = S_{\sigma_1\sigma_2,\tau_2}$$

$$S_{\sigma_1\sigma_2\sigma_1,\tau_2}S_{\sigma_1,\sigma_1} = S_{\sigma_1,\sigma_1}S_{\sigma_1\sigma_2\sigma_1,\tau_2}.$$

CASE $n = 3$.

LEMMA. From relation $r_4 = \sigma_1 \sigma_2 \tau_1 \sigma_2^{-1} \sigma_1^{-1} \tau_2^{-1}$ follows 3 non-trivial relations, applying which we can remove 3 generators:

$$S_{\sigma_1\sigma_2\sigma_1,\tau_1} = S_{\sigma_2,\tau_2}, \quad S_{\sigma_1\sigma_2,\tau_2} = S_{\sigma_1,\sigma_1} S_{\sigma_2\sigma_1,\tau_1} S^{-1}_{\sigma_1,\sigma_1},$$

$$S_{\sigma_1\sigma_2\sigma_1,\tau_2} = S_{\sigma_2\sigma_1,\sigma_1} S_{\sigma_2,\sigma_2} S_{\sigma_1,\tau_1} S^{-1}_{\sigma_2\sigma_1,\sigma_1} S^{-1}_{\sigma_1,\sigma_1} S^{-1}_{\sigma_2,\sigma_2} S_{\sigma_1,\sigma_1}.$$

CASE $n = 3$.

LEMMA.

From relation $r_5 = \sigma_2\sigma_1\tau_2\sigma_1^{-1}\sigma_2^{-1}\tau_1^{-1}$ follows relations:

$$(a_{13}a_{23})S_{\sigma_1,\tau_1} = S_{\sigma_1,\tau_1}(a_{13}a_{23}),$$

$$S_{\sigma_2\sigma_1,\tau_1}a_{13}^{-1} = a_{23}a_{12}(S_{\sigma_2\sigma_1,\tau_1}a_{13}^{-1})a_{12}^{-1}a_{23}^{-1},$$

$$a_{12}^{-1}S_{\sigma_2,\tau_2}a_{12} = a_{13}S_{\sigma_2,\tau_2}a_{13}^{-1},$$

$$a_{12}S_{\sigma_2\sigma_1,\tau_1}a_{12}^{-1} = a_{23}^{-1}S_{\sigma_2\sigma_1,\tau_1}a_{23}.$$

$$a_{12} = S_{\sigma_1,\sigma_1} = \sigma_1^2, \quad a_{13} = S_{\sigma_2\sigma_1,\sigma_1} = \sigma_2\sigma_1^2\sigma_2^{-1}, \quad a_{23} = S_{\sigma_2,\sigma_2} = \sigma_2^2,$$

$$b_{12} = S_{\sigma_1,\tau_1} = \sigma_1\tau_1, \quad b_{13} = S_{\sigma_2\sigma_1,\tau_1} = \sigma_2\sigma_1\tau_1\sigma_2^{-1}, \quad b_{23} = S_{\sigma_2,\tau_2} = \sigma_2\tau_2.$$

THE SINGULAR PURE BRAID GROUP

$$S_{1,\tau_1} = \tau_1 \cdot \sigma_1^{-1} = b_{12} a_{12}^{-1}, S_{1,\tau_2} = \tau_2 \cdot \sigma_2^{-1} = b_{23} a_{23}^{-1},$$
$$S_{\sigma_1,\sigma_1} = \sigma_1^2 = a_{12}, S_{\sigma_1,\tau_1} = \sigma_1 \tau_1 = b_{12},$$
$$S_{\sigma_1,\tau_2} = \sigma_1 \tau_2 \sigma_2^{-1} \sigma_1^{-1} = a_{23}^{-1} b_{13} a_{13}^{-1} a_{23}, S_{\sigma_2,\sigma_2} = \sigma_2^2 = a_{23},$$
$$S_{\sigma_2,\tau_1} = \sigma_2 \tau_1 \sigma_1^{-1} \sigma_2^{-1} = b_{13} a_{13}^{-1}, S_{\sigma_2,\tau_2} = \sigma_2 \tau_2 = b_{23},$$
$$S_{\sigma_1\sigma_2,\sigma_2} = \sigma_1 \sigma_2^2 \sigma_1^{-1} = a_{23}^{-1} a_{13} a_{23}, S_{\sigma_1\sigma_2,\tau_1} = \sigma_1 \sigma_2 \tau_1 \sigma_1^{-1} \sigma_2^{-1} \sigma_1^{-1} = b_{23} a_{23}^{-1},$$
$$S_{\sigma_1\sigma_2,\tau_2} = \sigma_1 \sigma_2 \tau_2 \sigma_1^{-1} = a_{23}^{-1} b_{13} a_{23}, S_{\sigma_2\sigma_1,\sigma_1} = \sigma_2 \sigma_1^2 \sigma_2^{-1} = a_{13},$$
$$S_{\sigma_2\sigma_1,\tau_1} = \sigma_2 \sigma_1 \tau_1 \sigma_2^{-1} = b_{13}, S_{\sigma_2\sigma_1,\tau_2} = \sigma_2 \sigma_1 \tau_2 \sigma_1^{-1} \sigma_2^{-1} \sigma_1^{-1} = b_{12} a_{12}^{-1},$$
$$S_{\sigma_1\sigma_2\sigma_1,\sigma_1} = \sigma_1 \sigma_2 \sigma_1^2 \sigma_2^{-1} \sigma_1^{-1} = a_{23}, S_{\sigma_1\sigma_2\sigma_1,\sigma_2} = \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1^{-1} \sigma_2^{-1} = a_{12},$$
$$S_{\sigma_1\sigma_2\sigma_1,\tau_1} = \sigma_1 \sigma_2 \sigma_1 \tau_1 \sigma_2^{-1} \sigma_1^{-1} = b_{23}, S_{\sigma_1\sigma_2\sigma_1,\tau_2} = \sigma_1 \sigma_2 \sigma_1 \tau_2 \sigma_1^{-1} \sigma_2^{-1} = b_{12}.$$

THE SINGULAR PURE BRAID GROUP

THEOREM.

The singular pure braid group SP_3 is generated by elements

$$a_{12}, \quad a_{13}, \quad a_{23}, \quad b_{12}, \quad b_{13}, \quad b_{23},$$

and is defined by relations:

$$a_{12}a_{13}a_{12}^{-1} = a_{23}^{-1}a_{13}a_{23}, \quad a_{12}a_{23}a_{12}^{-1} = a_{23}^{-1}a_{13}^{-1}a_{23}a_{13}a_{23},$$

$$a_{12}b_{12} = b_{12}a_{12}$$

$$a_{13}b_{13} = b_{13}a_{13},$$

$$a_{23}b_{23} = b_{23}a_{23},$$

$$b_{12}(a_{13}a_{23})b_{12}^{-1} = a_{13}a_{23},$$

$$a_{12}b_{13}a_{12}^{-1} = a_{23}^{-1}b_{13}a_{23},$$

$$a_{12}b_{23}a_{12}^{-1} = a_{23}^{-1}a_{13}^{-1}b_{23}a_{13}a_{23}.$$

THE SINGULAR PURE BRAID GROUP

COROLLARY.

$$a_{12}^{-1}a_{13}a_{12}=a_{13}a_{23}a_{13}a_{23}^{-1}a_{13}^{-1},\quad a_{12}^{-1}a_{23}a_{12}=a_{13}a_{23}a_{13}^{-1},$$
$$a_{12}^{-1}b_{13}a_{12}=a_{13}a_{23}b_{13}a_{23}^{-1}a_{13}^{-1},\quad a_{12}^{-1}b_{23}a_{12}=a_{13}b_{23}a_{13}^{-1}.$$

THE SINGULAR PURE BRAID GROUP

PROPOSITION. Generators of SG_3 act on the generators of SP_3 by the rules:

– action of σ_1: $a_{12}^{\sigma_1} = a_{12}, \quad a_{13}^{\sigma_1} = a_{13} a_{23} a_{13}^{-1}, \quad a_{23}^{\sigma_1} = a_{13};$

$$b_{12}^{\sigma_1} = b_{12}, \quad b_{13}^{\sigma_1} = a_{13} b_{23} a_{13}^{-1}, \quad b_{23}^{\sigma_1} = b_{13};$$

– action of σ_2: $a_{12}^{\sigma_2} = a_{23}^{-1} a_{13} a_{23}, \quad a_{13}^{\sigma_2} = a_{12}, \quad a_{23}^{\sigma_2} = a_{23};$

$$b_{12}^{\sigma_2} = a_{23}^{-1} b_{13} a_{23}, \quad b_{13}^{\sigma_2} = b_{12}, \quad b_{23}^{\sigma_2} = b_{23};$$

– action ofτ_1: $a_{12}^{\tau_1} = a_{12}, \quad a_{13}^{\tau_1} = b_{12}^{-1} a_{23} b_{12}, \quad a_{23}^{\tau_1} = b_{12}^{-1} a_{23}^{-1} a_{13} a_{23} b_{12},$

$$b_{12}^{\tau_1} = b_{12}, \quad b_{13}^{\tau_1} = b_{12}^{-1} b_{23} b_{12}, \quad b_{23}^{\tau_1} = b_{12}^{-1} a_{12} b_{13} a_{12}^{-1} b_{12},$$

–action of τ_2: $a_{12}^{\tau_2} = b_{23}^{-1} a_{13} b_{23}, \quad a_{13}^{\tau_2} = b_{23}^{-1} a_{23} a_{12} a_{23}^{-1} b_{23}, \quad a_{23}^{\tau_2} = a_{23},$

$$b_{12}^{\tau_2} = b_{23}^{-1} b_{13} b_{23}, \quad b_{13}^{\tau_2} = b_{23}^{-1} a_{23} b_{12} a_{23}^{-1} b_{23}, \quad b_{23}^{\tau_2} = b_{23}.$$

THE SINGULAR PURE BRAID GROUP

THEOREM. 1) $SP_3 = \widetilde{V}_3 \rightthreetimes \mathbb{Z}$, where $\mathbb{Z} = \langle a_{12} \rangle$;
2) $\widetilde{V}_3$ has a presentation

$$\widetilde{V}_3 = \langle a_{13}, a_{23}, b_{13}, b_{23}, b_{12} \mid\mid [a_{13}, b_{13}] = [a_{23}, b_{23}] = [a_{13}a_{23}, b_{12}] = 1 \rangle$$

and is an HNN-extension with base group V_3, stable letter b_{12} and associated subgroups $A \cong B = \langle a_{13}a_{23} \rangle$ and identity isomorphism $A \to B$:

$$\widetilde{V}_3 = \langle V_3, b_{12} \mid rel(V_3), \quad b_{12}^{-1}(a_{13}a_{23})b_{12} = a_{13}a_{23} \rangle,$$

where $rel(V_3)$ is the set of relations in V_3.
3) $V_3 = \langle a_{13}, a_{23}, b_{13}, b_{23} \mid\mid [a_{13}, b_{13}] = [a_{23}, b_{23}] = 1 \rangle \cong \mathbb{Z}^2 * \mathbb{Z}^2$.

THE SINGULAR PURE BRAID GROUP

QUESTION. Is it true that $Z(SG_n)$ is a direct factor in SP_n?

THE SINGULAR PURE BRAID GROUP

THEOREM. The singular pure braid group SP_3 is generated by elements

$$\delta, \quad a_{13}, \quad a_{23}, \quad b_{12}, \quad b_{13}, \quad b_{23},$$

and is defined by relations:

$$\delta b_{12} = b_{12}\delta, \quad \delta a_{13} = a_{13}\delta, \quad \delta a_{23} = a_{23}\delta, \quad \delta b_{13} = b_{13}\delta, \quad \delta b_{23} = b_{23}\delta.$$

$$a_{13} b_{13} = b_{13} a_{13}, \quad a_{23} b_{23} = b_{23} a_{23}, \quad b_{12}(a_{13} a_{23}) b_{12}^{-1} = a_{13} a_{23}.$$

COROLLARY. SP_3 is the direct product

$$SP_3 = Z \times \widetilde{V}_3,$$

where $Z = \langle \delta \rangle$ is the center of SP_3 and $\widetilde{V}_3 = \langle a_{13}, a_{23}, b_{12}, b_{13}, b_{23} \rangle$.

THANK
YOU