Base fields of csp-rings and cardinal characteristics of the continuum

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$\mathbf{P}$ is the set of all primes,
$\mathbf{N}$ is the set of all positive integers,
$\mathbf{Z}$ is the ring of integers,
$\mathbf{Q}$ is the field of rational numbers,
$\mathbf{R}$ is the real line.

Let $\chi = (k_p)_{p\in\mathbf{P}}$, where $k_p \in \mathbf{N}\cup\{0,\infty\}$ for all p, and let $L_\chi = \left\{p \in \mathbf{P} \;\middle|\; k_p \neq 0\right\}$. We set

$$R_p = \begin{cases} \text{the ring of } p\text{-adic integers} & \text{if } k_p = \infty; \\ \mathbf{Z}/p^{k_p}\mathbf{Z} & \text{if } k_p < \infty. \end{cases}$$

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Let the set $L = L_\chi$ be infinite. We introduce the notations

$$K_\chi = \prod_{p \in L} R_p, \qquad T_\chi = \bigoplus_{p \in L} R_p \subset K_\chi.$$

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Later P. A. Krylov introduced csp-rings as a generalization of rings of pseudorational numbers.

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The field R/T_χ as well as every field isomorphic to it is called a *base field* of the csp-ring R. Every such field (i.e., a field that can be embedded in K_χ/T_χ as a subring) has characteristic 0 and a cardinality not exceeding $2^{\aleph_0} = \mathfrak{c}$.

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Question. Which fields may serve as base fields of csp-rings?

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Thus every realization theorem for base fields of csp-rings can be considered as a realization theorem for endomorphism rings (in suitable categories).

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In view of this fact, when studying base fields of csp-rings, we can restrict ourselves to the case when χ contains only 0's and 1's:

$$K_L = \prod_{p\in L} \mathbf{Z}_p, \qquad T_L = \bigoplus_{p\in L} \mathbf{Z}_p \subset K_L,$$

where L is an infinite set of primes and $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$.

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Definition. An infinite subset $L \subset \mathbf{P}$ is called *universal* if every nonconstant polynomial from $\mathbf{Q}[x]$ can be factored in $\mathbf{Z}_p[x]$ into the product of polynomials of degree 1 for almost all $p \in L$.

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Remark. The concept of universal set will be unchanged if in its definition we replace $\mathbf{Z}_p$ with the ring of p-adic integers or with the p-adic number field.

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Cardinal characteristics of the continuum (which are more commonly used in set theory and topology) turned out to be a powerful tool for the study of base fields of csp-rings.

Cardinal characteristics of the continuum

Let $\langle \mathbf{N}^{\mathbf{N}}, \prec \rangle$ be the set of all functions $\mathbf{N} \to \mathbf{N}$, where

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Definition. We say that a set $E \subset \mathbf{N}^{\mathbf{N}}$ is

– *bounded* if there is $z \in \mathbf{N}^{\mathbf{N}}$ such that $z' \prec z$ for all $z' \in E$;

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$$\mathfrak{b} \stackrel{\text{def}}{=} \min\{|E| \mid E \text{ is an unbounded subset of } \mathbf{N}^{\mathbf{N}}\},$$

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$\mathfrak{b}$ and $\mathfrak{d}$ have the following properties:

$$\aleph_1 \leqslant \operatorname{cf}(\mathfrak{b}) = \mathfrak{b} \leqslant \operatorname{cf}(\mathfrak{d}) \leqslant \mathfrak{d} \leqslant \mathfrak{c}.$$

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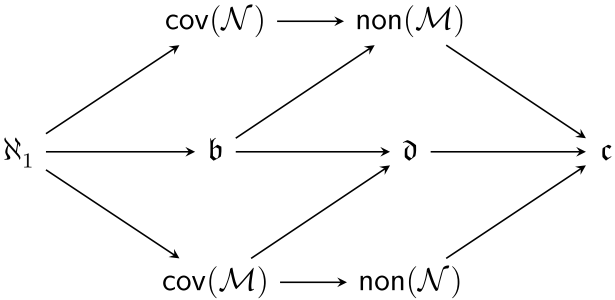
All inequalities between these cardinal characteristics are summarized in Cichoń's diagram.

Short version of Cichoń's diagram



(One goes from smaller to larger cardinals by moving along the arrows; all inequalities are non-strict.)

Remark. If we assign values $\aleph_1$ or $\aleph_2$ to all characteristics from this diagram and such an assignment does not contradict the diagram, then there is a model of ZFC realizing it.

Algebraically closed fields as base fields

Theorem 4. *Let* K_L/T_L *contain a subring* F *which is an algebraically closed field such that* $|F| < \mathfrak{b}$. *Then the natural inclusion* $F \to K_L/T_L$ *can be extended to an embedding of* $\overline{F(x)}$ *into* K_L/T_L, *where* $\overline{F(x)}$ *is the algebraic closure of the simple transcendental extension* $F(x)$ *of* F.

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If $|F| \leqslant \mathfrak{b}$ and $\operatorname{char} F = 0$, then F can be embedded in G.

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In particular, the conditions of Theorem 6 are equivalent if we assume the generalized continuum hypothesis, the continuum hypothesis or Martin's axiom since

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Martin's Axiom. If $\mathcal{X}$ is a compact Hausdorff space and there is no uncountable family of pairwise nonintersecting open subsets of $\mathcal{X}$, then $\mathcal{X}$ is not the union of less then $\mathfrak{c}$ nowhere dense subsets.

Characteristic $\mathfrak{ie}_L$

$$K_L = \prod_{p \in L} \mathbf{Z}_p$$

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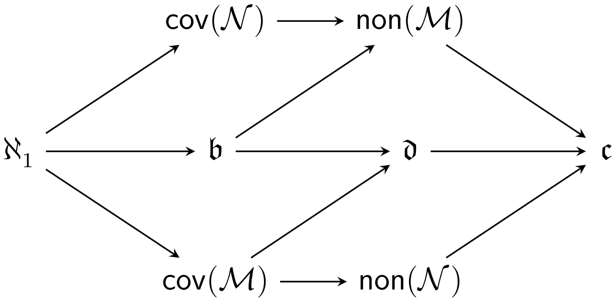
We introduce a new characteristic which depends on L:

let $\mathfrak{ie}_L$ (from the words "infinitely equal") denote the smallest cardinality of a set $B \subset K_L$ with the following property:

for any $b \in K_L$ there is $d \in B$ such that $b \approx d$.

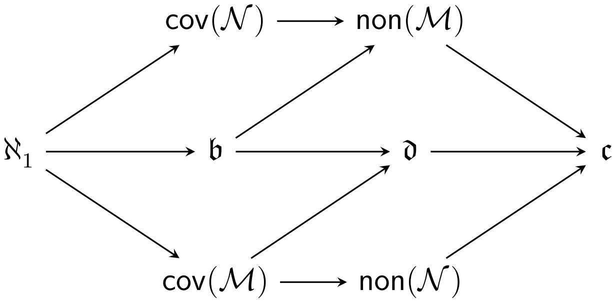


1. $\aleph_1 \leqslant \mathfrak{ie}_L \leqslant \mathsf{non}(\mathcal{M})$.

2. If $\sum_{p\in L} \frac{1}{p} < \infty$, then $\mathfrak{ie}_L \geqslant \mathsf{cov}(\mathcal{N})$.

3. (A. Blass) If $\sum_{p\in L} \frac{1}{p} = \infty$, then $\mathfrak{ie}_L \leqslant \mathsf{non}(\mathcal{N})$.

4. Martin's axiom implies that $\mathfrak{ie}_L = \mathfrak{c}$ for every $L \subset \mathbf{P}$.



Theorems 7 and 8 are related with the cardinal $\max(\mathfrak{ie}_L, \mathfrak{b})$ which will appear in Theorem 9.

Theorem 7. *If* $\mathfrak{d} = \mathfrak{b}$, *then* $\sup_L \max(\mathfrak{ie}_L, \mathfrak{b}) = \mathsf{non}(\mathcal{M})$.

Theorem 8. *Each of the following inequalities is consistent with* ZFC:

(a) $\mathfrak{ie}_L > \mathfrak{b}$;

(b) $\mathfrak{ie}_L < \mathfrak{b}$;

(c) $\max(\mathfrak{ie}_L, \mathfrak{b}) > \max(\mathfrak{ie}_X, \mathfrak{b})$.

Purely transcendental extensions of $\mathbf{Q}$

Theorem 9. *Suppose K_L/T_L contains a field F such that $|F| < \max(\mathfrak{ie}_L, \mathfrak{b})$. Then the natural inclusion $F \to K_L/T_L$ can be extended to an embedding $F(x) \to K_L/T_L$ with $F(x)$ being the simple transcendental extension of F.*

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Theorem 11. *Let L be a universal set, and let F be some countable field such that* $\operatorname{char} F = 0$ *and* $F \not\cong \mathbf{Q}$. *Then the set of* csp-*rings* $R \subset K_L$ *with* $R/T_L \cong F$ *has cardinality* $\mathfrak{c}$ *(all such rings are pairwise nonisomorphic).*

Some literature on cardinal characteristics of continuum:

T. Bartoszyński, H. Judah. *Set theory*: *on the structure of the real line*. Wellesley: A. K. Peters, 1995.

A. Blass. *Combinatorial cardinal characteristics of the continuum // Handbook of set theory*. Dordrecht et al.: Springer, 2010, p. 395–489.

E. K. van Douwen. *The integers and topology // Handbook of set-theoretic topology*. Amsterdam et al.: North-Holland, 1984, p. 111–167.

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Thank you for your attention.

E. A. Timoshenko. *Base fields of* csp-*rings* // Algebra and Logic, 2010, **49**:4, p. 378–385.

E. A. Timoshenko. *Purely transcendental extensions of the field of rational numbers as base fields of* csp-*rings* // Tomsk State University Journal of Mathematics and Mechanics, 2013, **5(25)**, p. 30–39 (in Russian).

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