# Linear and uniformly continuous surjections between $C_p$-spaces

Dedicated to S. Gul'ko on the occasion of his 75th birthday

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(jointly with A. Eysen and A. Leiderman)

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It is my pleasure to present a talk at the conference dedicated to Prof. Sergey Gul'ko. I have rectly the opportunity to study Gul'ko's paper about preservation of the dimension by uniform homeomorphisms between $C_p$-spaces. The technique which was developed by Gul'ko in that paper, and specially the so called Gul'ko supports' are one of the most interesting and helpful achievemnts in $C_p$-theory.

# Motivation

The $C_p$-theory was introduced by Arhangel'skii and his students (recall that for a Tychonoff space $X$ the set of all continuous functions on $X$ with the pointwise convergence topology is denoted by $C_p(X)$).

One of the main directions in $C_p$-theory is the investigation of properties $\mathcal{P}$ such that if $X \in \mathcal{P}$ and $C_p(X)$ is linearly or uniformly homeomorphic to $C_p(Y)$, then $Y \in \mathcal{P}$. Probably, the best results in that direction are Pestov's theorem, stating that if $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic, then $\dim X = \dim Y$, and Uspenskii's theorem that pseudocompactness and compactness are determined by the uniform structure of $C_p$-spaces.

Let's note that if we consider the function spaces with the uniform convergence, the Pestov's result is not anymore true. Indeed, according to classical Milyutin's theorem if $X$ and $Y$ are uncountable metric compacta, then their function spaces $C(X)$ and $C(Y)$ equipped with the sup metric are linearly homeomorphic.

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Pestov's result was generalized by Gul'ko who proved that $\dim X = \dim Y$ providing $C_p(X)$ and $C_p(Y)$ are uniformly homeomorphic (recall that a map $T : C_p(X) \longrightarrow C_p(Y)$ is uniformly continuous if for every neighborhood $U$ of $0_Y$ there is a nbd $V$ of $0_X$ such that $T(f) - T(g) \in U$ provided $f - g \in V$).

Gul'ko's result motivated the investigation of properties $\mathcal{P}$ such that if $X \in \mathcal{P}$ and $C_p(X)$ is uniformly homeomorphic to $C_p(Y)$, then $Y \in \mathcal{P}$.

Another direction in the $C_p$-theory is to investigate properties $\mathcal{P}$ such that if $X \in \mathcal{P}$ and there is a linear continuous (or uniformly continuous) surjection $T : C_p(X) \longrightarrow C_p(Y)$, then $Y \in \mathcal{P}$. For example, in the class of metrizable spaces completeness is preserved by linear continuous surjections (Baars-de Groot-Pelant), while other absolute Borel classes are preserved by uniformly continuous surjections (Marciszewski-Pelant). Moreover, absolute Borel classes greater than 2 and all projective classes are preserved by homeomorphisms between $C_p(X)$ and $C_p(Y)$ when $X, Y$ are metrizable (Marciszewski).

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On the other hand, Leiderma-Levin-Pestov proved that the Arhangelskii question has a positive answer in dimension 0 when $X$ and $Y$ are metric compacta. The last result was extended for arbitrary compact spaces by Kawamura-Leiderman.

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# Main results

**Theorem 1 [Eysen-V]**

If there is a linear continuous surjection $T : C_p(X) \to C_p(Y)$, then $\dim X = 0$ implies $\dim Y = 0$.

We consider properties $\mathcal{P}$ of metric spaces such that:

(a) if $X \in \mathcal{P}$ and $F \subset X$ is closed, then $F \in \mathcal{P}$;

(b) $\mathcal{P}$ is closed under finite products;

(c) if $X$ is a countable union of closed subsets each having the property $\mathcal{P}$, then $X \in \mathcal{P}$;

(d) if $f : X \to Y$ is a perfect map with countable fibers and $Y \in \mathcal{P}$, then $X \in \mathcal{P}$;

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For example, 0-*dimensionality*, *countable-dimensionality* and *strongly countable-dimensionality* satisfy conditions $(a) - (d)$, while 0-dimensionality and countable-dimensionality satisfy also conditions $(b) - (e)$.

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In the class of metrizable spaces, Theorem 1 has a stronger analogue:

**Theorem 2 [Eysen-Leiderman-V]**

Let $T : D_p(X) \longrightarrow D_p(Y)$ be a linear continuous surjection and $\mathcal{P}$ be a topological property satisfying either conditions $(a) - (d)$ or $(b) - (e)$. If $X$ is a metric space and $Y$ is perfectly normal, then $Y$ has the property $\mathcal{P}$ provided $X \in \mathcal{P}$.

Here $D_p(X)$ denote either $C_p(X)$ or $C_p^*(X)$, where $C_p^*(X)$ is the set of bounded continuous functions with the pointwise topology.

We say that a surjection $T : D_p(X) \longrightarrow D_p(Y)$ is *inversely bounded* if for every norm bounded sequence $\{g_n\} \subset C^*(Y)$ there is a norm bounded sequence $\{f_n\} \subset C^*(X)$ with $T(f_n) = g_n$ for each $n$. The following notion was introduced by Gartside-Feng: A map $T : D_p(X) \longrightarrow D_p(Y)$ is $c$-good if for every $g \in C^*(Y)$ there is $f \in C^*(X)$ with $||f|| \leq c.||g||$. Note that every linearly continuous surjection $T : C_p^*(X) \longrightarrow C_p^*(Y)$ is inversely bounded, as wee as every $c$-good map.

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## Theorem 3 [Eysen-V]

Let $T : D_p(X) \longrightarrow D_p(Y)$ be a c-good uniformly continuous surjection. Then $Y$ is 0-dimensional provided so is $X$.

## Corollary [Eysen-V]

Let $T : C_p^*(X) \longrightarrow D_p(Y)$ be a linear continuous surjection. Then $Y$ is 0-dimensional provided so is $X$.

Theorem 3 has a stronger version in case $X$ is metrizable and $Y$ is perfectly normal:

## Theorem 4 [Eysen-Leiderman-V]

Let $T : D_p(X) \longrightarrow D_p(Y)$ be an inversely bounded uniformly continuous surjection and $\mathcal{P}$ be a topological property satisfying either conditions $(a) - (d)$ or $(b) - (e)$, where $X$ is metrizable and $Y$ is perfectly normal. Then $Y$ has the property $\mathcal{P}$ provided $X \in \mathcal{P}$.

In particular, Theorem 3 is true if $\mathcal{P}$ is countable-dimensional or strongly countable-dimensional.

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# Proofs

## Proof of Theorem 3

A subset $E(X) \subset C(X)$ is called a $QS$-algebra (Gul'ko) if:
(i) $f+g$, $f \cdot g$ and $\lambda.f$ belong to $E(X)$ provided $f, g \in E(X)$ and $\lambda$ is a rational number;
(ii) For every $x \in X$ there is a its nbd $U$ there exists $f \in E(X)$ such that $f(x) = 1$ and $f(X \backslash U) = 0$.

Following Gul'ko, the proof is reduced to the following proposition:

## Proposition 1

Let $\overline{X}$ and $\overline{Y}$ be metric compactifications of $X$ and $Y$, and $H \subset \overline{X}$ be a $\sigma$-compact space containing $X$. Suppose $E(H)$ is a $QS$-algebra on $H$, $E(X) = \{\overline{f}|X : \overline{f} \in E(H)\}$ and $E(Y) \subset C(Y)$ is a family such that every $g \in E(Y)$ is extendable to a map $\overline{g} : \overline{Y} \longrightarrow \mathbb{R}$ and $E(\overline{Y}) = \{\overline{g} : g \in E(Y)\}$ containing a $QS$-algebra $\Gamma$ on $\overline{Y}$. Let also $\varphi : E_p(X) \longrightarrow E_p(Y)$ be an uniformly continuous surjection which is inversely bounded.
If $H$ has a property $\mathcal{P}$ satisfying conditions $(a) - (d)$, then there exists a $\sigma$-compact set $Y_\infty \subset \overline{Y}$ containing $Y$ with $Y_\infty \in \mathcal{P}$.

# Proofs

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# Proof of Proposition 1

For every $\bar{f} \in E(H)$ denote by $f$ the restriction $\bar{f}|X$. For every $y \in \overline{Y}$ there is a map $\alpha_y : E(H) \longrightarrow \overline{\mathbb{R}}$, $\alpha_y(\bar{f}) = \overline{\varphi(f)}(y)$. Since $\varphi$ is uniformly continuous, so is each $\alpha_y|E_p(X)$, $y \in Y$.
Suppose $H = \bigcup_k H_k$ is the union of an increasing sequence $\{H_k\}$ of compact sets.
We use the idea of supports introduced by Gul'ko and the extension of that notion introduced by Mikolaj Krupski. For every $y \in \overline{Y}$ and every $p, k \in \mathbb{N}$ we define the families
$\mathcal{A}^k(y) = \{K \subset H_k : K \text{ is closed and } a(y, K) < \infty\}$ and
$\mathcal{A}^k_p(y) = \{K \subset H_k : K \text{ is closed and } a(y, K) \leq p\}$, where
$a(y, K) = \sup\{|\alpha_y(\bar{f}) - \alpha_y(\bar{g})| : \bar{f}, \bar{g} \in E(H), |\bar{f}(x) - \bar{g}(x)| < 1 \ \forall x \in K\}$.

Possibly, some or both of the values $\alpha_y(\bar{f}), \alpha_y(\bar{g})$ from the definition of $a(y, K)$ could be $\pm\infty$. That's why we use the following agreements:

(*) $\infty + \infty = \infty, \infty - \infty = -\infty + \infty = 0, -\infty - \infty = -\infty.$

Note that $a(y, \varnothing) = \infty$ since $\varphi$ is surjective.

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For every $\overline{f} \in E(H)$ denote by $f$ the restriction $\overline{f}|X$. For every $y \in \overline{Y}$ there is a map $\alpha_y : E(H) \longrightarrow \overline{\mathbb{R}}$, $\alpha_y(\overline{f}) = \overline{\varphi(f)}(y)$. Since $\varphi$ is uniformly continuous, so is each $\alpha_y|E_p(X)$, $y \in Y$.
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We use the idea of supports introduced by Gul'ko and the extension of that notion introduced by Mikolaj Krupski. For every $y \in \overline{Y}$ and every $p, k \in \mathbb{N}$ we define the families
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(1) For every $y \in Y$ there is $p, k \in \mathbb{N}$ such that $\mathcal{A}_p^k(y)$ contains a finite nonempty subset of $X$.

(2) Each set $Y_p^k = \{y \in \overline{Y} : \mathcal{A}_p^k(y) \neq \varnothing\}$ is a closed subset of $\overline{Y}$.

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(7) For every $q$ let $[H_k]^q$ denote the set of all $q$-points subsets of $H_k$ endowed with the Vietoris topology. The map $\Phi_{kpq} : M^k(p,q) \longrightarrow [H_k]^q$, $\Phi_{kpq}(y) = K_{kp}(y)$, is continuous.

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We can complete the proof of Proposition 1. Suppose $H$ has a property $\mathcal{P}$ satisfying conditions $(a)-(d)$. Then so does $H^q_k$ for each $k,q$ because $H_k$ is closed in $H$. But $[H_k]^q$ is homeomorphic to the open subset $W_q = \{(x_1,..,x_q) \in H^q_k : x_i \neq x_j\}$ of $H^q_k$. So, $W_q$ has the property $\mathcal{P}$ as a countable union of closed subsets of $H^q_k$. Hence, each set $\Phi^n_{kpq}(F^k_n(p,q))$ also has the property $\mathcal{P}$ because it is a compact subset of $W_q$. Finally, since the maps $\Phi^n_{kpq} : F^k_n(p,q) \longrightarrow \Phi^n_{kpq}(F^k_n(p,q))$ are perfect and have finite fibers, each $F^k_n(p,q)$ has the property $\mathcal{P}$. Therefore, $Y_\infty = \bigcup\{F^k_n(p,q) : n,p,q,k = 1,2,..\}$ has the property $\mathcal{P}$.

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We can finish the proof of Theorem 3. Let $T : D_p(X) \longrightarrow D_p(Y)$ be a uniformly continuous inversely bounded surjection. It suffices to show that for every map $h : Y \longrightarrow Z$, where $Z$ is separable metric space, there are maps $h_0 : Y \longrightarrow Y_0$ and $g : Y_0 \longrightarrow Z$ such that $\dim Y_0 = 0$ and $h = g \circ h_0$ (when $h = g \circ h_0$ for some map $g$, we write $h_0 \succ h$). We fix such $h$ and let $\bar{h} : \beta Y \longrightarrow \overline{Z}$ be a continuous extension of $h$, where $\overline{Z}$ is a compact metric space. For every $\Psi \subset C(\beta X)$ we denote by $\triangle\Psi$ the diagonal product of all functions from $\Psi$. Clearly, $\triangle\Psi(\beta X)$ is a subset of the product $\prod\{\mathbb{R}_f : f \in \Psi\}$, and let $\pi_f : \triangle\Psi(\beta X) \longrightarrow \mathbb{R}_f$ be the projection. Following Gul'ko, we call a set $\Psi \subset C(\beta X)$ admissible if the family $\pi(\Psi) = \{\pi_f : f \in \Psi\}$ is a $QS$-algebra on $\triangle\Psi(\beta X)$.

We construct by induction two sequences $\{\Psi_n\}_{n\geq 1} \subset C(\beta X)$ and $\{\Phi_n\}_{n\geq 1} \subset C(\beta Y, \overline{\mathbb{R}})$ of countable sets, countable $QS$-algebras $\Lambda_n$ on $Y'_n = (\triangle\Phi'_n)(\beta Y)$, where $\Phi'_n = \{\overline{T(f)} : \bar{f} \in \Psi_n\}$, satisfying the following conditions for every $n \geq 1$. Here, $\overline{T(f)} : \beta Y \longrightarrow \overline{\mathbb{R}}$ is the extension of $T(f)$.

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(3.5) $\Phi_1 \subset C(\beta Y)$ is admissible and $\triangle\Phi_1 \succ \overline{h}$;

(3.6) $\Phi_n \subset \Phi_{n+1} = \Phi'_n \cup \{\lambda \circ (\triangle\Phi'_n) : \lambda \in \Lambda_n\}$;

(3.7) Each $\Psi_n$ is admissible, $\dim(\triangle\Psi_n)(\beta X) = 0$ and $\Psi_n \subset \Psi_{n+1}$;

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