On multivalued groups of order 3

(based on joint work with Jin Guo and Ilia Ponomarenko)

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Intro

The concept of multivalued groups goes back to a construction in the Buchstaber-Novikov paper (1971).

At present, the theory of multivalued groups is an actively developing part of mathematics and finds numerous applications in various fields:

- topology (Buchstaber, Kholodov, ...)
- dynamical systems (Buchstaber, Veselov, Gaifullin, ...)
- Hopf algebras (Buchstaber, Rees, ...)
- discrete mathematics (Buchstaber, Vershik, Yagodovskii, ...)

Recent developments:

- classification of involutive 2-valued groups (Buchstaber, Veselov, Gaifullin, 2019–2024)
- theory of cyclic n-valued groups and its applications (Buchstaber, Vesnin, 2024)

Definition of multivalued groups

Let X be a set, n a positive integer.

$\mathfrak{X} = (X, \cdot, \star)$ is n-valued group on X with the operations of multiplication $\cdot$ and of taking inverse $\star$, if

1. $x \cdot y = [(x \cdot y)_1, \ldots, (x \cdot y)_n]$ is a multiset of elements of X
2. the multiplication is associative, that is $[((x \cdot y)_i \cdot z)_j : \ 1 \leq i, j \leq n] = [(x \cdot (y \cdot z)_i)_j : \ 1 \leq i, j \leq n]$
3. there is an identity element $e \in X$ such that for all $x \in X$, $(x \cdot e)_i = (e \cdot x)_i = x, \quad i = 1, \ldots, n$
4. $\star$ maps X to X so that e belongs to $x \cdot x^\star$ and $x^\star \cdot x$.

$|X|$ is the order of $\mathfrak{X}$ and $m_{x,y}^z$ is the multiplicity of z in $x \cdot y$.

A bijection $f : X \to X'$, $x \mapsto x'$, is an isomorphism of $\mathfrak{X}$ onto an n'-valued group $\mathfrak{X}' = (X', \cdot, \star)$, if for all $x, y, z \in X$,

$$\frac{m_{x,y}^z}{n} = \frac{m_{x',y'}^{z'}}{n'}.$$

Involutive multivalued groups

A multivalued group $\mathfrak{X} = (X, \cdot, \star)$ is involutive, if the taking inverse $\star$ is an involution on X satisfying the following

1. $e \in x \cdot y \implies y = x^\star$ for all $x, y \in X$
2. $m(x) = m(x^*)$, where $m(x) = m^e_{x,x}$
3. $(x \cdot y)^\star = y^\star \cdot x^\star$ for all $x, y \in X$.

Buchshtaber, Vershik, Evdokimov, Ponomarenko (1996):
The categories of involutive multivalued groups and complex combinatorial algebras are equivalent.

The definition of an involutive multivalued group is weaker than that in: Buchshtaber, Veselov, Gaifullin, Rus. Math. Surveys (2022).

Coset multivalued groups

Let G and A be (ordinary) groups, and let A act on G via a homomorphism $\varphi : A \to \mathrm{Aut}(G)$. Then one can define a coset n-valued group $\mathfrak{X} = \mathfrak{X}(G, A, \varphi)$ (or $\mathfrak{X}(G, A)$) by setting

- $n = |A|$
- $X = G/\varphi(A)$ is the set of the orbits of the action of A on G
- for all $x, y, z \in X$,

$$m^z_{x,y} = |\{a \in A : g \cdot h^a \in z\}|, \quad x^* = \{(g^{-1})^a : a \in A\},$$

  where $g \in x$ and $h \in y$.

A coset multivalued group is always involutive.

Example. Let $G = V$ be a linear space over $\mathbb{F}_q$ and $A = \mathrm{GL}(V)$. Then $\mathfrak{X}$ is isomorphic to $(|V| - 1)$-group on $X = \{e, x\}$, where

$$e = \{\bar{0}\}, \; x = V \setminus \{\bar{0}\}, \; \text{and } x \cdot x = [e, x, \ldots, x].$$

Every coset n-group of order 2 is isomorphic to $\mathfrak{X}$ of this form.

Involutive multivalued groups of order 3

Let $\mathfrak{X}$ be an involutive n-valued group on X of order 3.

Our goals are

1. to give necessary conditions on multiplicities of $\mathfrak{X}$;
2. to present a construction of $\mathfrak{X}$ via strongly regular graphs;
3. to classify all coset multivalued groups $\mathfrak{X}$.

Clearly, for $X = \{e, x, y\}$ exactly one of the following hold:

(i) $x^\star = y$ and $y^\star = x$;

(ii) $x^\star = x$ and $y^\star = y$.

Parameters of involutive multivalued groups of order 3: (i)

Theorem 1(i)

Let $\mathfrak{X}$ be an involutive n-valued group on $X = \{e, x, y\}$ of order 3, and $a(x) = m^x_{x,x}$. If $x^\star = y$ and $y^\star = x$, then

$$x \cdot x = [\underbrace{x, \ldots, x}_{a(x)}, \underbrace{y, \ldots, y}_{n-a(x)}], \qquad y \cdot y = [\underbrace{x, \ldots, x}_{n-a(x)}, \underbrace{y, \ldots, y}_{a(x)}],$$

$$x \cdot y = y \cdot x = [\underbrace{e, \ldots, e}_{n-2a(x)}, \underbrace{x, \ldots, x}_{a(x)}, \underbrace{y, \ldots, y}_{a(x)}];$$

Let $\mathfrak{X}_n(a)$ with $a = a(x)$ be the n-valued involutive group of order 3, defined in Theorem 1(i).

The multivalued groups $\mathfrak{X}_n(a)$ and $\mathfrak{X}_{n'}(a')$ are isomorphic if and only if $a/n = a'/n'$.

Buchshtaber, Vershik, Evdokimov, Ponomarenko (1996):
$\mathfrak{X}_{2k+1}(k)$ is an involutive $(2k+1)$-valued group with

$$x \cdot x = [\underbrace{x, \dots, x}_{k}, \underbrace{y, \dots, y}_{k+1}], \quad y \cdot y = [\underbrace{x, \dots, x}_{k+1}, \underbrace{y, \dots, y}_{k}],$$

$$x \cdot y = y \cdot x = [e, \underbrace{x, \dots, x}_{k}, \underbrace{y, \dots, y}_{k}].$$

Evdokimov, Ponomarenko (...2024):
$\mathfrak{X} = \mathfrak{X}_{2k+1}(k)$ is coset if and only if $4k+3$ is a prime power.
$\mathfrak{X} = \mathfrak{X}(G, A)$, where $G = \mathbb{F}_q^+$ and $A = (\mathbb{F}_q^\times)^2$ for $q = 4k+3$.

Main Theorem (i)

A multivalued group $\mathfrak{X}$ of order 3 with $x^\star = y$ and $y^\star = x$ is a coset group if and only if $\mathfrak{X} \simeq \mathfrak{X}_{2k+1}(k)$, where $4k+3$ is a prime power.

Parameters of involutive multivalued groups of order 3: (ii)

Theorem 1(ii)

Let $\mathfrak{X}$ be an involutive n-valued group on $X = \{e, x, y\}$ of order 3, and $m(x) = m^e_{x,x}$, $m(y) = m^e_{y,y}$, $a(x) = m^x_{x,x}$. If $x^\star = x$ and $y^\star = y$, then

$$x{\cdot}x = [\underbrace{e,\dots,e}_{m(x)},\underbrace{x,\dots,x}_{a(x)},\underbrace{y,\dots,y}_{n-m(x)-a(x)}],\ y{\cdot}y = [\underbrace{e,\dots,e}_{m(y)},\underbrace{x,\dots,x}_{a(y)},\underbrace{y,\dots,y}_{n-m(y)-a(y)}],$$

$$x \cdot y = y \cdot x = [\underbrace{x,\dots,x}_{a(x,y)},\underbrace{y,\dots,y}_{n-a(x,y)}],$$

where $a(x,y) = r(n - m(x) - a(x))$, $a(y) = ra(x,y)$ for $r = m(y)/m(x)$.

Let $\mathfrak{X}_n(m_1, m_2, a)$ with $m_1 = m(x)$, $m_2 = m(y)$ and $a = a(x)$, be the n-valued involutive group of order 3, defined in Theorem 1(ii).

The multivalued groups $\mathfrak{X}_n(m_1, m_2, a)$ and $\mathfrak{X}_{n'}(m'_1, m'_2, a')$ are isomorphic if and only if $u/n = u'/n'$ for each $u \in \{m_1, m_2, a\}$.

Strongly regular graphs (SRG)

A graph Γ is strongly regular, if it is not complete or edgeless, and the number of common neighbors of two vertices α and β in Γ

$$= \begin{cases} k & \text{if } \alpha = \beta \\ \lambda & \text{if } \alpha \sim \beta \\ \mu & \text{if } \alpha \not\sim \beta \end{cases}$$

The integers $k > 0, \lambda, \mu \geq 0$ (called the parameters of Γ) satisfy

$$k(k - 1 - \lambda) = (v - k - 1)\mu, \qquad (*)$$

where v is the number of vertices of Γ.

Relation $(*) \implies$ we can exclude v as a parameter of Γ.

The complement $\overline{\Gamma}$ of Γ is a strongly regular graph with parameters $\overline{k} = v - k - 1$, $\overline{\lambda} = v - 2k + \mu - 2$, and $\overline{\mu} = v - 2k - \lambda$.

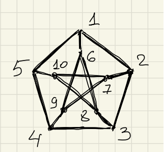
Multivalued groups of order 3 via SRG

Theorem 2

Let Γ be a strongly regular graph with parameters (k, λ, μ), and let n be the least common multiple of k and $\overline{k}$. Then there is an n-valued involutive group $\mathfrak{X}(\Gamma) = \mathfrak{X}(k, \lambda, \mu)$ on the set $\{e, x, y\}$, such that $x^\star = x$, $y^\star = y$, and

$$m(x) = m^e_{x,x} = \frac{n}{k}, \quad m(y) = m^e_{y,y} = \frac{n}{\overline{k}}, \quad a(x) = m^x_{x,x} = \frac{n\lambda}{k}.$$

Example. The Petersen graph on 10 vertices is strongly regular with parameters $(3, 0, 1)$. Since $\overline{k} = v - k - 1 = 6$, we have $\mathfrak{X}(3, 0, 1)$ is the 6-valued group on 3-element set $\{e, x, y\}$ with



$$x \cdot x = [e, e, y, y, y, y]$$
$$y \cdot y = [e, x, y, y, y, y]$$
$$x \cdot y = y \cdot x = [x, x, y, y, y, y].$$

Remark 1. The multivalued group $\mathfrak{X}(\Gamma)$ defined by SRG Γ depends on the parameters of Γ only. Therefore, any two nonisomorphic SRG with the same parameters define the same multivalued group, e. g., the Hamming graph $H(2,4)$ and the Shrikhande graph on 16 vertices with parameters $(6,2,2)$ induce the unique multivalued group $\mathfrak{X}_{18}(3,2,6)$ of order 3. Note that for infinitely many positive integers v there can be exponentially many nonisomorphic SRG with v vertices and the same parameters.

Remark 2. Not each multivalued involutive group of order 3 is of the form $\mathfrak{X}(\Gamma)$ for some SRG Γ. Indeed, for each natural number $q = 4\ell + 1$, there exists a multivalued group $\mathfrak{X} = \mathfrak{X}_{2\ell}(1,1,\ell - 1)$. Assume that $\mathfrak{X} = \mathfrak{X}(\Gamma)$ for some SRG Γ. Then one can find that the parameters v and k of Γ are equal to $4\ell + 1$ and 2ℓ, respectively. It follows that $\overline{k} = 2\ell = k$. This is possible only if v is a sum of two squares. Thus $\mathfrak{X} = \mathfrak{X}(\Gamma)$ only if $4\ell + 1$ is a sum of two squares. In particular, $\mathfrak{X} = \mathfrak{X}_{10}(1,1,4)$ cannot be defined via SRG.

At present, neither a classification of all SRG, nor a description of their attainable parameters are known.

However, there is a classification of rank 3 graphs, the subclass of SRG, and the description of the their attainable parameters. Details can be found in:

Brouwer, A. E., Van Maldeghem, H.: Strongly regular graphs. Cambridge University Press, Cambridge (2022)

We show that this classification makes possible to describe the coset multivalued groups of order 3.

Rank 3 groups and graphs

$K \leq \mathrm{Sym}(\Omega)$, a 2-orbit or orbital is an orbit of K on $\Omega \times \Omega$.
Rank of K = the number of 2-orbits.

An orbital graph Γ of K is a graph with the vertex set Ω having one of the orbitals of K as the arc (or edge) set.

Let K have rank 3.
Then one of the 2-orbits is the diagonal of $\Omega \times \Omega$, so K is transitive.

$|K|$ is odd $\implies$ two nondiagonal 2-orbits of K are mutual transposes and the corresponding orbital graphs are tournaments, they are called Paley tournaments corresponding to K.

$|K|$ is even $\implies$ two nondiagonal 2-orbits of K are symmetric and yield two complementary SRG, they are called rank 3 graphs corresponding to G.

Example. The Petersen graph is the orbital graph of $K = \mathrm{Alt}(5)$ acting naturally on the set Ω of 2-element subsets of $\{1, \ldots, 5\}$.

Considering the coset group $\mathfrak{X}(G, A)$, we may suppose that $A = \varphi(A)$ acts on G faithfully that is $A \leq \mathrm{Aut}(G) \leq \mathrm{Sym}(G)$.

$K = G \rtimes A$ acts on G: $x \mapsto x^k = x^a \cdot g$ for $x \in G$, $k = (g, a) \in K$.

Note that G is a normal regular subgroup of $K = G \rtimes A \leq \mathrm{Sym}(G)$ and A is the stabilizer of the identity element of G (regular = acting transitively and fixed-point-freely).

$X = G/\varphi(A)$ is of size 3 $\Longleftrightarrow$ $K = G \rtimes A \leq \mathrm{Sym}(G)$ is of rank 3.

Theorem 3

Let $K = G \rtimes A \leq \mathrm{Sym}(G)$ be a group of rank 3 of even order and Γ is the orbital rank 3 graph of K, then the multivalued groups $\mathfrak{X}(\Gamma)$ and $\mathfrak{X}(G, A)$ are isomorphic.

It can be deduced from the classification of rank 3 groups that such a group K with a normal regular subgroup G must be affine.

We say that $K = G \rtimes A \leq \mathrm{AGL}_d(p)$ is affine, if

- $G = V$ is a linear space over a prime field $\mathbb{F}_p$
- $A = K_0 \leq \mathrm{GL}(V) = \mathrm{GL}_d(p)$
- K acts on V by maps of the form $x \mapsto Ax + b$, where $A \in \mathrm{GL}_d(p)$, $b \in V$.

Example. The pentagon graph and the Petersen graph are both rank 3 graphs, but only the former is affine.

Sometimes it is convenient to enlarge the field (if possible):

- $G = V$ is a linear space over a field $\mathbb{F}_q$, $|V| = q^a = p^d$.
- $K = V \rtimes A \leq \mathrm{A\Gamma L}(V) = \mathrm{A\Gamma L}_a(q)$, where $A \leq \Gamma\mathrm{L}(V) = \Gamma\mathrm{L}_a(q) = \mathrm{GL}_a(q) \rtimes \mathrm{Aut}(\mathbb{F}_q)$
- K acts on V by maps of the form $x \mapsto Ax^{\phi} + b$, where $A \in \mathrm{GL}_a(q)$, $\phi \in \mathrm{Aut}(\mathbb{F}_q)$, $b \in V$.

Liebeck's classification of affine rank 3 groups (1987)

$K = V \rtimes A$ is a primitive affine group, $|V| = p^d$.

(A) Infinite classes. These are:

(1) $A \leq \Gamma L_1(p^d)$;
(2) A is imprimitive as a linear group;
(3) A stabilizes the decomposition of $V \simeq \mathbb{F}_q^{2m}$ into $V = V_1 \otimes V_2$, where $p^d = q^{2m}$, dim $V_1 = 2$ and dim $V_2 = m$;
(4) $A \trianglerighteq SL_m(\sqrt{q})$ and $p^d = q^m$, where 2 divides $\frac{d}{m}$;
(5) $A \trianglerighteq SL_2(\sqrt[3]{q})$ and $p^d = q^2$, where 3 divides $\frac{d}{2}$;
(6) $A \trianglerighteq SU_m(q)$ and $p^d = q^{2m}$;
(7) $A \trianglerighteq \Omega_{2m}^{\pm}(q)$ and $p^d = q^{2m}$;
(8) $A \trianglerighteq SL_5(q)$ and $p^d = q^{10}$;
(9) $A \trianglerighteq B_3(q)$ and $p^d = q^8$;
(10) $A \trianglerighteq D_5(q)$ and $p^d = q^{16}$;
(11) $A \trianglerighteq Sz(q)$ and $p^d = q^4$.

(B) 'Extraspecial' groups (finitely many).

(C) 'Exceptional' groups (finitely many).

The disadvantage: many groups, not so many graphs...

Skresanov's description (2021)

G is from class **(A)**, Γ is an orbital graph of G.

(1) $G \leq \mathrm{A\Gamma L}_1(q)$. Then either $\mathrm{Aut}(\Gamma) \leq \mathrm{A\Gamma L}_1(q)$, or $q \leq 89^2$.

(2) $G \leq \mathrm{A\Gamma L}_{2m}(q)$ preserves $H_q(2,m)$, $m \geq 3$. Then

$$\mathrm{Aut}(\Gamma) = \mathbb{F}^{2m} \rtimes ((\mathrm{GL}_2(q) \circ \mathrm{GL}_m(q)) \rtimes \mathrm{Aut}(\mathbb{F})).$$

(3) $G \leq \mathrm{A\Gamma L}_{10}(q)$ preserves $A(5,q)$. Then

$$\mathrm{Aut}(\Gamma) = \mathbb{F}^{10} \rtimes ((\Gamma\mathrm{L}_5(q)/\{\pm 1\}) \times (\mathbb{F}^\times/(\mathbb{F}^\times)^2)).$$

(4) $G \leq \mathrm{A\Gamma L}_{2m}(q)$ preserves $\mathrm{VO}^\epsilon_{2m}(q)$, $m \geq 2$, $\epsilon = \pm$. Then

$$\mathrm{Aut}(\Gamma) = \mathbb{F}^{2m} \rtimes \Gamma\mathrm{O}^\epsilon_{2m}(q).$$

(5) $G \leq \mathrm{A\Gamma L}_4(q)$ preserves $\mathrm{VSz}(q)$, $q = 2^{2e+1}$, $e \geq 1$. Then

$$\mathrm{Aut}(\Gamma) = \mathbb{F}^4 \rtimes ((\mathbb{F}^\times \times \mathrm{Sz}(q)) \rtimes \mathrm{Aut}(\mathbb{F})).$$

(6) $G \leq \mathrm{A\Gamma L}_{16}(q)$ preserves $\mathrm{VD}_{5,5}(q)$. Then

$$\mathrm{Aut}(\Gamma) = \mathbb{F}^{16} \rtimes ((\mathbb{F}^\times \circ \mathrm{Inndiag}(D_5(q))) \rtimes \mathrm{Aut}(\mathbb{F})).$$

The automorphism groups for small affine graphs of rank 3

Thus, if G is an affine group of rank 3 and Γ is an orbital graph of G, then either $\mathrm{Aut}(\Gamma)$ or

1. $G \leq \mathrm{A\Gamma L}_1(q)$, $q \leq 89^2$;
2. G is from class **(B)**;
3. G is from class **(C)**.

In particular, $\mathrm{Aut}(\Gamma)$ is known with a finite number of exceptions.

Guo, Vasil'ev, Wang (2024):

$\mathrm{Aut}(\Gamma)$ is known in the remaining cases.

Corollary

If Γ is a graph of rank 3, then $\mathrm{Aut}(\Gamma)$ is known.

Main Theorem (short version)

A multivalued group $\mathfrak{X}$ of order 3 is a coset group if and only if $\mathfrak{X} \simeq \mathfrak{X}_{2k+1}(k)$, where $4k+3$ is a prime power, or a multivalued group $\mathfrak{X}(k, \lambda, \mu)$, where the integers k, λ, μ are the parameters of an affine rank 3 graph (described in the separate statement).

The description includes all sets (k, λ, μ) of attainable parameters (without repetitions), and for every of set (k, λ, μ), possible groups G and A with $\mathfrak{X}(k, \lambda, \mu) = \mathfrak{X}(G, A)$.

Guo J., Vasil'ev A. V., Ponomarenko I.: On multivalued groups of order 3, Science China Math., in press (2025), doi:10.1007/s11425-024-2388-0, also arXiv:2410.04341

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