Growth of multivalued dynamics

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n-valued groups

Consider the n-th symmetric power of a set: $\mathrm{Sym}^n(X) = X^n/S_n$.

An n-valued multiplication structure on X is a map

$$*\colon X \times X \to \mathrm{Sym}^n(X); \quad x * y = [z_1, \ldots, z_n] \in \mathrm{Sym}^n(X).$$

It naturally extends to a map

$$* : X \times \mathrm{Sym}^n(X) \to \mathrm{Sym}^{n^2}(X); \quad x * [z_1, \ldots, z_n] = x * z_1 \cup \ldots \cup x * z_n$$

n-valued groups

Definition

A set X equipped with an n-valued multiplication $*$ is called an *n-valued group* if it has the following properties

- Associativity: the n^2-sets $(x * y) * z$ and $x * (y * z)$ are equal for all $x, y, z \in X$.
- Unit: $e \in X$ such that for all $x \in X$ we have

$$e * x = x * e = [x, \ldots, x].$$

- Inverse: a map $\mathrm{inv} : X \to X$ such that

$$e \in x * \mathrm{inv}(x) \text{ and } e \in \mathrm{inv}(x) * x.$$

Examples

Example 1

Let G be a (1-valued) group. Consider $x * y = [xy, xy, \dots, xy]$ and $\mathrm{inv}(g) = g^{-1}$.

Example 2

$X = \mathbb{N} \cup \{0\}$ with $e = 0$, $\mathrm{inv}(x) = x$ and the multiplication

$$x * y = \big[x + y, |x - y|\big].$$

Note that in order to verify associativity one has to check that

$$\big[x+y+z, |x-y-z|, x+|y-z|, |x-|y-z||\big] = \big[x+y+z, |x+y-z|, |x-y|+z, ||x-y|-z|\big].$$

Coset groups

Let G be a (1-valued) group and $A \subset \operatorname{Aut} G$ a finite group of automorphisms of order n.

Denote $X = G/A$, and denote by $\pi : G \to X$ the quotient map.

Now define the n-valued multiplication by the formula

$$\pi(x) * \pi(y) = \Big[\pi\big(xy^a\big)\Big| a \in A\Big].$$

Note that the multiplication does not depend on the specific choice of x and y.

Theorem

The set X with multiplication $*$, the unit $e = e_G$ and the inverse $\operatorname{inv}(\pi(x)) = \pi(x^{-1})$ is an n-valued group.

Coset groups

Example 3

Let $G = \mathbb{Z}$ and $A = \{\varepsilon, -\varepsilon\}$, where $x^{-\varepsilon} = -x$. Then G/A can be associated with non-negative integers, and the multiplication is given by the formula

$$|x| * |y| = [|x + y^{\varepsilon}|, |x + y^{-\varepsilon}|] = \Big[|x| + |y|, \big||x| - |y|\big|\Big],$$

which gives us a 2-valued group isomorphic to the one in Example 2.

The same (up to isomorphism) 2-valued group can also be obtained by taking

$$G = \langle a, b \mid a^2 = e, b^2 = e \rangle$$

with A swapping a and b.

Cyclic dynamics

Let X be an n-valued group and $z \in X$. Consider the action of z on symmetric powers of X by right multiplication:

$$T\colon \operatorname{Sym}^k(X) \to \operatorname{Sym}^{kn}(X); \quad T(y) = y * z.$$

Consider the map $\operatorname{Set}\colon \operatorname{Sym}(X) \to 2^X$ that maps an unordered tuple of elements of X into a subset of X consisting of all unique elements of that tuple. For $r \in \mathbb{N}$ and $y \in X$ denote by $\xi_y(r)$ the number of unique elements in $T^r(y)$:

$$\xi_y(r) = \left|\operatorname{Set}\left(T^r(y)\right)\right|.$$

The function $\xi_y\colon \mathbb{N} \to \mathbb{N}$ is called *the growth function* of the cyclic dynamic T in the point y.

Cayley graph

Suppose that X is an n-valued group generated (as a monoid) by a set S. Then the Cayley graph $\Gamma_{X,S} = \Gamma(V, E)$ is an oriented graph with the set of vertices $V = X$ and two vertices $u, v \in V$ are connected by an edge $e \in E$ if for some $s \in S$ we have $v \in u * s$.

Question

Is it true that if S is finite, then any vertice of the graph $\Gamma_{X,S}$ is a terminal vertex of only finitely many edges?

From now on, we only consider finitely generated groups.

Growth function

Even though we are not considering the Cayley graph $\Gamma = \Gamma_{X,S}$ as a metric space, we can still define a ball of radius $r \geqslant 0$ with the center in $x \in X$ as the set

$$B(x,r) = \left\{ y \in X \mid \exists m \in \mathbb{N}, m \leqslant r, \exists s_{i_1}, \dots s_{i_m} \in S, y \in \mathrm{Set}\left(x * s_{i_1} * s_{i_2} *, \dots, * s_{i_m}\right) \right\},$$

and a sphere

$$S(x,r) = B(x,r) \setminus B(x,r-1).$$

Then the *growth function* of X with respect to $x \in X$ is a function $\gamma_{\Gamma,x} \colon \mathbb{N} \to \mathbb{N}$, $\gamma_{\Gamma,x}(r) = |B(x,r)|$.

In this framework, the growth function of an n-valued dynamic of an element can be viewed as a growth function of a cyclic semigroup generated by that element.

Growth function

Two nondecreasing positive functions $f(r)$ and $g(r)$ have the same growth if there is a constant $C > 0$ so that for all $r \geqslant 0$ we have

$$f(r/C) \leqslant g(r) \leqslant f(Cr).$$

Theorem 1

Let X be an n-valued group with generating sets S and S'. Let $y, y' \in X$. Then the functions $\gamma_{\Gamma_{X,S,y}}$ and $\gamma_{\Gamma_{X,S',y'}}$ have the same growth.

Theorem 2

Let $X = G/A$ be an n-valued coset group. Then the growth function of G/A has the same growth function as the group G.

Growth of cyclic dynamics

Let M be an monoid with a generating set L. Any element $g \in M$ can be represented as a word over L. Denote by $l(g)$ the minimal length of such a word. For any non-negative r let $S(r)$ be the set of all elements g in M such that $l(g) = r$, and $B(r)$ the set of all elements g in M such that $l(g) \leqslant r$.

Theorem 3

Let $X = G/A$ be an n-valued coset group and $z \in X$. Then for the dynamic T, generated by z, the growth function ξ_y satisfies the following inequality:

$$\frac{1}{n}|S(r)| \leqslant \xi_y(r) \leqslant |B(r)|,$$

where $M = \langle z^a : a \in A \rangle \subset G$.

Cyclically presented groups

Let w be a word on $x_1, \ldots, x_n$ and let θ be the map:

$$\theta\colon x_i \mapsto x_{i+1},\ x_n \mapsto x_1.$$

Let $G = \langle x_1, \ldots, x_n \mid w, \theta(w), \ldots, \theta^{n-1}(w) \rangle$. The map θ induces a cyclic subgroup of order n in $\mathrm{Aut}(G)$. Let X be the coset group G/A.

In their 2024 paper, Vesnin and Buchstaber pose a question on growth in such n-valued group. The growth functions of this group and of the dynamic of its generating element can be studied using theorems 2 and 3.

A special case of 2-valued groups

Theorem 4

Let X be a 2-valued group such that $\mathrm{inv}(x) = x$ for every $x \in X$. Then for any $z \in X$ the growth function ξ_y satisfies the inequality

$$\xi_y(r) \leqslant r(r+1).$$

Thank you for your attention!