First integrals of Hamiltonian systems on 2-surfaces

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Introduction

Poisson bracket and Hamiltonian systems

Let M be a smooth manifold, dim $M = N$.
Let $f, g \in C^\infty(M)$.

In local coordinates $y = (y^1, \dots, y^N)$ on M *the Poisson bracket* is given by:

$$h^{ij}(y) = \{y^i, y^j\}, \qquad \{f, g\} = h^{ij}(y) \frac{\partial f(y)}{\partial y^i} \frac{\partial g(y)}{\partial y^j}, \qquad i, j = 1, \dots, N.$$

Poisson bracket allows to define a Hamiltonian system on M:

$$\frac{d}{dt} y^i = \{y^i, H(y)\}, \qquad i = 1, \dots, N.$$

Poisson bracket and Hamiltonian systems

In canonical coordinates $(y^1, \dots, y^N) = (x^1, \dots, x^n, p_1, \dots, p_n)$, $N = 2n$ we have

$$\{x^i, p_j\} = \delta^i_j, \qquad \{x^i, x^j\} = 0, \qquad \{p_i, p_j\} = 0, \qquad i, j = 1, \dots, n;$$

$$\{F, H\} = \sum_{j=1}^{n} \left(\frac{\partial F}{\partial x^j} \frac{\partial H}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial H}{\partial x^j} \right).$$

Canonical Hamiltonian equations:

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x^i}.$$

The first integrals $F = F(y)$ of this system satisfy the following condition:

$$\dot{F} = \{F, H\} = 0.$$

Integrable geodesic flow on a 2-surface

Let

$$ds^2 = g_{ij}(x)dx^i dx^j, \qquad i,j = 1,2$$

be a Riemannian metric on $\mathbb{M}^2$. The geodesic flow is called *integrable* if the Hamiltonian system

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x^i}, \qquad H = \frac{1}{2} g^{ij} p_i p_j$$

possesses an additional first integral $F : T^*\mathbb{M}^2 \to \mathbb{R}$ such that

$$\dot{F} = \{F, H\} = \sum_{j=1}^{2} \left(\frac{\partial F}{\partial x^j} \frac{\partial H}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial H}{\partial x^j} \right) = 0$$

and F is functionally independent with H almost everywhere.

Global integrability of Hamiltonian systems

Topological obstacles to the complete integrability

Theorem (V.V. Kozlov)
If a genus of a surface $\mathbb{M}^2$ *is different from 0 or 1 (that is* $\mathbb{M}^2$ *is homeomorphic neither to a sphere* $\mathbb{S}^2$ *nor to a torus* $\mathbb{T}^2$*), then the geodesic flow of any analytical Riemannian metric on this surface has no first integral which is analytical on* $T^*\mathbb{M}^2$ *and independent on the Hamiltonian.*

Polynomial in momenta first integrals

It is known that there exist metrics of two types on the 2-torus with an integrable geodesic flow, namely:

$$ds^2 = \Lambda(x)(dx^2 + dy^2), \qquad F_1 = p_2,$$

$$ds^2 = (\Lambda_1(x) + \Lambda_2(y))(dx^2 + dy^2), \qquad F_2 = \frac{\Lambda_2 p_1^2 - \Lambda_1 p_2^2}{\Lambda_1 + \Lambda_2}.$$

Conjecture about degrees of polynomial first integrals (V.V. Kozlov).
The maximal degree of any *irreducible* polynomial in momenta first integral of geodesic flow on a surface of genus g seems to be not larger than $4 - 2g$.

Cubic first integral

Choose the conformal coordinates (x, y), such that $ds^2 = \Lambda(x,y)(dx^2 + dy^2)$.

$$H = \frac{p_1^2 + p_2^2}{2\Lambda}, \quad F = a_0(x,y)p_1^3 + a_1(x,y)p_1^2 p_2 + a_2(x,y)p_1 p_2^2 + a_3(x,y)p_2^3.$$

The following relations on the metrics and coefficients of the first integral hold:

$$a_2 - a_0 = c_0, \quad a_3 - a_1 = c_1,$$

where $c_0, c_1 \in \mathbb{R}$ are Kolokoltsov constants; moreover,

$$a_1\Lambda_y + 2\Lambda a_{0x} + 3a_0\Lambda_x = 0,$$

$$3a_1\Lambda_y + 2\Lambda a_{1y} + (1 + a_0)\Lambda_x = 0.$$

$$(1 + a_0)\Lambda_y + \Lambda\left(a_{0y} + a_{1x}\right) + a_1\Lambda_x = 0,$$

It can be written in the following form:

$$\begin{pmatrix} 3a_0 & 2\Lambda & 0 \\ 1 + a_0 & 0 & 0 \\ a_1 & 0 & \Lambda \end{pmatrix} \begin{pmatrix} \Lambda \\ a_0 \\ a_1 \end{pmatrix}_x + \begin{pmatrix} a_1 & 0 & 0 \\ 3a_1 & 0 & 2\Lambda \\ 1 + a_0 & \Lambda & 0 \end{pmatrix} \begin{pmatrix} \Lambda \\ a_0 \\ a_1 \end{pmatrix}_y = 0.$$

Integrable geodesic flow on the 2-torus

Theorem (N.V. Denisova, V.V. Kozlov)
Suppose that the geodesic flow on the 2-torus admits a homogeneous in momenta first integral F_n which is independent on the Hamiltonian. Suppose that
1) either F_n is even on p_1, p_2
2) or F_n is even on $p_1(p_2)$ and odd on $p_2(p_1)$,
then there exists an additional polynomial in momenta first integral of degree ≤ 2.

Theorem (N.V. Denisova, V.V. Kozlov)
Suppose that the geodesic flow on the 2-torus admits a homogeneous in momenta first integral F_n which is independent on the Hamiltonian. The metric $\Lambda(x, y)$ is assumed to be a trigonometric polynomial. Then there exists an additional polynomial in momenta first integral of degree ≤ 2.

Integrable geodesic flow on the 2-torus

Theorem (M. Bialy, A.E. Mironov)
If the Hamiltonian system has an integral F which is a homogeneous polynomial of degree n, then on the covering plane $\mathbb{R}^2$ there exist the global semi-geodesic coordinates (t,x) such that

$$ds^2 = g^2(t,x)dt^2 + dx^2, \qquad H = \frac{1}{2}\left(\frac{p_1^2}{g^2} + p_2^2\right)$$

and F can be written in the form:

$$F_n = \sum_{k=0}^{n} \frac{a_k(t,x)}{g^{n-k}} p_1^{n-k} p_2^k.$$

Here the last two coefficients can be normalized by the following way:

$$a_{n-1} = g, \; a_n = 1.$$

Integrable geodesic flow on the 2-torus

The condition $\{F, H\} = 0$ *is equivalent to the quasi-linear PDEs*

$$U_t + A(U)U_x = 0, \tag{1}$$

where $U^T = (a_0, \dots, a_{n-1}),\ a_{n-1} = g,$

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & a_1 \\ a_{n-1} & 0 & \dots & 0 & 0 & 2a_2 - na_0 \\ 0 & a_{n-1} & \dots & 0 & 0 & 3a_3 - (n-1)a_1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{n-1} & 0 & (n-1)a_{n-1} - 3a_{n-3} \\ 0 & 0 & \dots & 0 & a_{n-1} & na_n - 2a_{n-2} \end{pmatrix}.$$

Quasi-linear system of PDEs

Quasi-linear systems of the form

$$A(U)U_x + B(U)U_y = 0,$$

$$U_t = A(U)U_x, \qquad U = (u_1, \dots, u_n)^T$$

appears in such areas like

- gas-dynamics
- non-linear elasticity
- integrable geodesic flows on 2-torus

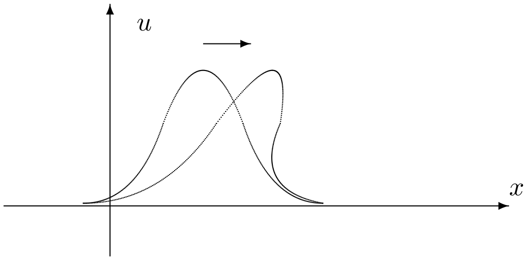
and many others.

Hopf equation (inviscid Burgers' equation)

Consider the following equation $u_t + uu_x = 0$. The solution of the Cauchy problem $u|_{t=0} = g(x)$ is given by the implicit formula

$$u(x,t) = g(x - ut).$$

It follows from this formula that the higher any point is placed, the faster it is.



Semi-Hamiltonian systems

Theorem (M. Bialy, A.E. Mironov)
(1) is semi-Hamiltonian system. Namely, there is a regular change of variables

$$U \mapsto (G_1(U), \dots, G_n(U))$$

such that for some $F_1(U), \dots, F_n(U)$ the following conservation laws hold:

$$(G_i(U))_x + (F_i(U))_y = 0, \qquad i = 1, \dots, n.$$

Moreover, in the hyperbolic domain, where eigenvalues $\lambda_1, \dots, \lambda_n$ of A(U) are real and pairwise distinct, there exists a change of variables

$$U \mapsto (r_1(U), \dots, r_n(U))$$

such that the system can be written in Riemannian invariants:

$$(r_i)_x + \lambda_i(r)(r_i)_y = 0, \qquad i = 1, \dots, n.$$

Semi-Hamiltonian systems

The eigenvalues of a semi-Hamiltonian system $\lambda_i(r)$ satisfy the following relations:

$$\partial_{r_j}\frac{\partial_{r_i}\lambda_k}{\lambda_i-\lambda_k}=\partial_{r_i}\frac{\partial_{r_j}\lambda_k}{\lambda_j-\lambda_k},\qquad i\neq j\neq k\neq i.$$

It means that there exists a diagonal metrics

$$ds^2=H_1^2(r)dr_1^2+\ldots+H_N^2(r)dr_N^2,$$

with Christoffel symbols satisfying the following relations

$$\Gamma^k_{ki}=\frac{\partial_{r_i}\lambda_k}{\lambda_i-\lambda_k},\qquad i\neq k.$$

S.P. Tsarev: the generalized hodograph method.

Natural mechanical systems and the Maupertuis principle

Let $\mathbb{M}^n$ be a smooth manifold with the Riemannian metric $ds^2 = g_{ij}dx^i dx^j$. Consider a Hamiltonian system

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i}, \quad H = \frac{1}{2}g^{ij}(x)p_i p_j + V(x), \quad i,j = 1,\ldots,n,$$

where $V(x)$ is a smooth potential. Define

$$Q^{2n-1} = \{H(x,p) = h, h > maxV(x)\}.$$

Construct a new Hamiltonian

$$\widetilde{H} = \frac{1}{2}\frac{g^{ij}(x)p_i p_j}{h - V(x)}$$

such that $\widetilde{H} = 1$ on Q^{2n-1}. $\widetilde{H}$ corresponds to the new metric

$$\widetilde{g_{ij}} = (h - V(x))g_{ij}.$$

Natural mechanical systems and the Maupertuis principle

So we have
$$Q^{2n-1} = \{H(x,p) = h\} = \{\widetilde{H}(x,p) = 1\}.$$
It follows from here that trajectories of these two Hamiltonian systems coincide (up to a parametrization).

Suppose that the initial natural mechanical system (with H as a Hamiltonian) admits a first integral $f(x,p)$ on a fixed energy level Q^{2n-1}. Then the geodesic flow (with $\widetilde{H}$ as a Hamiltonian) admits a first integral $\widetilde{f}(x,p) = f(x, \frac{p}{|p|})$ on the whole $T^*\mathbb{M}^n$ (except maybe a zero energy level) and $f\,|_{Q^{2n-1}} = \widetilde{f}\,|_{Q^{2n-1}}$.

Natural mechanical systems on the 2-torus

Consider a Hamiltonian system with the Hamiltonian

$$H = \frac{p_1^2 + p_2^2}{2} + V(x_1, x_2),$$

where V is assumed to be periodic function on the plane $\mathbb{R}^2$ with a period lattice $\Lambda \subset \mathbb{R}^2$.

1) If

$$V(x_1, x_2) = V(\alpha x_1 + \beta x_2),$$

where $\alpha, \beta \in \mathbb{R}$, then there exists a polynomial integral $F_1 = \alpha p_2 - \beta p_1$.

2) If

$$V(x_1, x_2) = V_1(\alpha_1 x_1 + \beta_1 x_2) + V_2(\alpha_2 x_1 + \beta_2 x_2),$$

where $\alpha_i, \beta_i \in \mathbb{R}$ are constants compatible with the period lattice Λ, then there exists a polynomial integral

$F_2 = (d_1 + d_2)p_1^2 + 4p_1p_2 - (d_1 + d_2)p_2^2 + 2(d_1 - d_2)(V_1 - V_2)$, $d_i = \alpha_i/\beta_i$.

Polynomial integrals of natural mechanical systems

- **3 degree**
  M. Bialy
  N.V. Denisova, V.V. Kozlov
- **4 degree**
  N.V. Denisova, V.V. Kozlov, D.V. Treschev
- **5 degree**
  A.E. Mironov
- **Higher degrees**
  Open problem

Magnetic geodesic flow (systems with gyroscopic forces)

$$\frac{d}{dt}y^i = \{y^i, H(y)\}_{mg}, \qquad i = 1, \ldots, N.$$

In coordinates $(y^1, \ldots, y^N) = (x^1, \ldots, x^n, p_1, \ldots, p_n)$, $N = 2n$ magnetic Poisson bracket is given by

$$\{x^i, p_j\}_{mg} = \delta^i_j, \qquad \{x^i, x^j\}_{mg} = 0, \qquad \{p_i, p_j\}_{mg} = \Omega_{ij}(x),$$

Consider a Hamiltonian system

$$\dot{x}^j = \{x^j, H\}_{mg}, \qquad \dot{p}_j = \{p_j, H\}_{mg}, \qquad j = 1, 2$$

on the 2-torus in presence of a magnetic field with $H = \frac{1}{2}g^{ij}p_i p_j$ and the Poisson bracket:

$$\{F, H\}_{mg} = \sum_{i=1}^{2}\left(\frac{\partial F}{\partial x^i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial x^i}\right) + \Omega(x^1, x^2)\left(\frac{\partial F}{\partial p_1}\frac{\partial H}{\partial p_2} - \frac{\partial F}{\partial p_2}\frac{\partial H}{\partial p_1}\right).$$

The only known examples of integrable geodesic flows on the 2-torus on all energy levels

Integrable geodesic flow

$$ds^2 = \Lambda(y)(dx^2 + dy^2), \qquad F_1 = p_1;$$

$$ds^2 = (\Lambda_1(x) + \Lambda_2(y))(dx^2 + dy^2), \qquad F_2 = \frac{\Lambda_2 p_1^2 - \Lambda_1 p_2^2}{\Lambda_1 + \Lambda_2}.$$

Integrable magnetic geodesic flow

$$ds^2 = dx^2 + dy^2, \quad \omega = B dx \wedge dy, \quad B = const \neq 0, \quad F_1 = \cos\left(\frac{p_1}{B} - y\right);$$

$$ds^2 = \Lambda(y)(dx^2 + dy^2), \qquad \omega = -u'(y) dx \wedge dy, \qquad F_1 = p_1 + u(y).$$

Magnetic geodesic flow and its integrability

Theorem (S.V. Bolotin, V.V. Ten)
Let $H = \frac{p_1^2+p_2^2}{2}$ *and the magnetic form* $\omega = \lambda(x,y)dx \wedge dy$*. The magnetic geodesic flow possesses an additional polynomial first integral iff the Fourier spectrum of* $\lambda(x,y)$ *lies on a straight line going through the origin and the average of* $\lambda(x,y)$ *over the whole torus is equal to 0.*

Consequence (S.V. Bolotin, V.V. Ten)
The degree of any irreducible polynomial first integral of such magnetic geodesic flow is equal to 1.

Quadratic first integrals on several energy levels

$$H = \frac{p_1^2 + p_2^2}{2\Lambda(x^1, x^2)}, \quad \dot{x}^j = \{x^j, H\}_{mg}, \quad \dot{p}_j = \{p_j, H\}_{mg}, \qquad j = 1, 2.$$

Theorem (A., Bialy, Mironov)
Consider the magnetic flow of the Riemannian metric $ds^2 = \Lambda(x, y)(dx^2 + dy^2)$ with the non-zero magnetic form ω. Suppose the magnetic flow admits a first integral F_2 on all energy levels such that F_2 is quadratic in momenta. Then in some coordinates we have

$$ds^2 = \Lambda(y)(dx^2 + dy^2), \qquad \omega = -u'(y)dx \wedge dy$$

so there exists another integral F_1 which is linear in momenta: $F_1 = p_1 + u(y)$, and F_2 can be written as a combination of H and F_1.

I.A. Taimanov: There is no additional irreducible quadratic first integral with analytic periodic coefficients even on 2 different energy levels!

Integrals of higher degrees on several energy levels

Lemma (A., Valyuzhenich) *Suppose that the geodesic flow on the 2-torus in a non-zero magnetic field admits an additional polynomial in momenta first integral* F *of an arbitrary degree* N *on* $\frac{N+1}{2}$ *or* $\frac{N+2}{2}$ *different energy levels* $\{H = E_1\}, \{H = E_2\} \dots$*. Then* F *is the first integral of the same flow on all energy levels.*

Theorem (A., Valyuzhenich) *Suppose that the geodesic flow on the 2-torus in a non-zero magnetic field admits an additional polynomial in momenta first integral* F *of an arbitrary degree* N *with analytic periodic coefficients on* $\frac{N+1}{2}$ *or* $\frac{N+2}{2}$ *different energy levels* $\{H = E_1\}, \{H = E_2\} \dots$*. Then the magnetic field and the metric are functions of one variable and there exists a linear in momenta first integral* F_1 *on all energy levels.*

Quadratic first integrals on a fixed energy levels

For a Riemannian metric $ds^2 = \Lambda(x, y)(dx^2 + dy^2)$ and quadratic in momenta first integral on the 2-torus on a fixed energy level we obtain the following system

$$A(U)U_x + B(U)U_y = 0,$$

where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ f & 0 & \Lambda & 0 \\ 2 & 1 & 0 & \frac{g}{2} \\ 0 & 0 & 0 & -\frac{f}{2} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & -\Lambda \\ 0 & 0 & -\frac{g}{2} & 0 \\ 2 & -1 & \frac{f}{2} & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} \Lambda \\ u_0 \\ f \\ g \end{pmatrix}.$$

Magnetic field has the form: $\qquad \Omega = \frac{1}{4}(g_x - f_y).$

M. Bialy, A.E. Mironov: This system is proved to be semi-Hamiltonian.

Dorizzi B., Grammaticos B., Ramani A. and Winternitz P.:
two commuting functions $\tilde{H}, \tilde{F}$ with respect to the standard Poisson bracket $\{\ ,\}$ were found by the following construction:

$$\tilde{H} = \frac{1}{2}\left(p_1 + R^{'}(y)\right)^2 + \frac{1}{2}\left(p_2 - S^{'}(x)\right)^2 + h,$$

$$\tilde{F} = \frac{1}{2}\left(p_2 - S^{'}(x)\right)^2 + R^{'}(y)\left(p_1 + R^{'}(y)\right) + S^{'}(x)\left(p_2 - S^{'}(x)\right) + f.$$

Here functions h and f are defined as follows:

$$h = \frac{1}{2}(S^{'})^2 + \frac{1}{2}(R^{'})^2 + SR^{''} + RS^{''} + \mu_2 - \mu_1, \qquad f = \frac{1}{2}(S^{'})^2 + SR^{''} + \mu_2,$$

where

$$\mu_1 = (S^{'})^2 + \frac{1}{2}\beta_2 S^2 - \beta_3 S, \qquad \mu_2 = -(R^{'})^2 - \frac{1}{2}\beta_1 R^2 + \beta_3 R.$$

Here functions $S(x), R(y)$ have to satisfy the following equations

$$S^{''} = \alpha S^2 + \beta_1 S + \gamma_1, \qquad R^{''} = -\alpha R^2 + \beta_2 R + \gamma_2,$$

$\alpha, \beta_j, \gamma_k$ are constants. These constants have to be chosen so that there are smooth periodic solutions S, R of these equations.

Commuting functions $\tilde{H}, \tilde{F}$ determine two new functions

$$H = \frac{p_1^2 + p_2^2}{2} + h, \qquad F = \frac{p_2^2}{2} + R^{'} p_1 + S^{'} p_2 + f$$

which are commuting with respect to the magnetic Poisson bracket and the magnetic field is

$$\Omega(x, y) = S^{''}(x) + R^{''}(y).$$

By Maupertuis' principle, one can modify H to give explicit examples of integrable magnetic geodesic flows on one energy level.

Example

The functions

$$H_E = \frac{p_1^2 + p_2^2}{2(E - h)}, \qquad F_2 = \frac{1}{2} p_2^2 + R^{'}(y) p_1 + S^{'}(x) p_2 + f$$

commute with respect to $\{\ \}_{mg}$ on the energy level $\{H_E = 1\}$. Notice that for any $E > \max h$, H_E is a perfectly defined Hamiltonian of the magnetic geodesic flow on the torus which has a quadratic integral F_2 on the energy level.

The only known explicit non-trivial solution

Dorizzi B., Grammaticos B., Ramani A. and Winternitz P.:

$$A(U)U_x + B(U)U_y = 0, \qquad U = (\Lambda, u_0, f, g)^T, \qquad \Omega = \frac{1}{4}(g_x - f_y).$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ f & 0 & \Lambda & 0 \\ 2 & 1 & 0 & \frac{g}{2} \\ 0 & 0 & 0 & -\frac{f}{2} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & -\Lambda \\ 0 & 0 & -\frac{g}{2} & 0 \\ 2 & -1 & \frac{f}{2} & 0 \end{pmatrix}.$$

Explicit solution:

$$U_0(x,y) = \begin{pmatrix} \Lambda(x,y) \\ u_0(x,y) \\ f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} 2E - 2h(x,y) \\ -8q(x,y) - 4(E - h(x,y)) \\ -4R'(y) \\ 4S'(x) \end{pmatrix}, \quad \Omega = S''(x) + R''(y),$$

$$h(x,y) = \frac{1}{2}(S')^2 + \frac{1}{2}(R')^2 + SR'' + RS'' + \mu_1 - \mu_2, \quad q(x,y) = \frac{1}{2}(S')^2 + SR'' + \mu_2,$$

here $\mu_1(x,y) = (S')^2 + \frac{1}{2}\beta_2 S^2 - \beta_3 S, \quad \mu_2(x,y) = -(R')^2 - \frac{1}{2}\beta_1 R^2 - \beta_3 R$ and

$$S'' = \alpha S^2 + \beta_1 S + \gamma_1, \quad R'' = -\alpha R^2 + \beta_2 R + \gamma_2.$$

Quadratic first integrals on a fixed energy level

Theorem (A., Bialy, Mironov)
There exist real analytic Riemannian metrics on the 2-torus which are arbitrary close to the Liouville metrics (and different from them) and a non-zero analytic magnetic fields such that magnetic geodesic flows on the energy level $\{H = \frac{1}{2}\}$ *have polynomial in momenta first integral of degree two.*

Crucial construction

Introduce the following evolution equations:

$$U_\tau = A_1(U)U_x + B_1(U)U_y,$$

where

$$A_1 = \begin{pmatrix} g & 0 & 0 & \Lambda \\ -2g & g & 0 & -2\Lambda \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} f & 0 & \Lambda & 0 \\ 2f & f & 2\Lambda & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This system defines the symmetry of the previous system so that this flow transforms solutions to solutions.

Crucial construction

One can easily check that

$$U_0(x,y) = \begin{pmatrix} \Lambda_1(x) + \Lambda_2(y) \\ 2\Lambda_2(y) - 2\Lambda_1(x) \\ 0 \\ 0 \end{pmatrix}$$

is the solution, where $\Lambda_1(x)$ and $\Lambda_2(y)$ are periodic positive functions: $\Lambda_1(x+1) = \Lambda_1(x),\ \Lambda_2(y+1) = \Lambda_2(y)$. This solution corresponds to the geodesic flow of the Liouville metric with zero magnetic field having the quadratic first integral of the form

$$F_2 = \frac{\Lambda_2(y)p_1^2 - \Lambda_1(x)p_2^2}{\Lambda_1(x) + \Lambda_2(y)}.$$

Λ_1 and Λ_2 are assumed to be real analytic periodic functions.

Local integrability of Hamiltonian systems

Polynomial integrals of geodesic flow

Choose the conformal coordinates (x, y), such that

$$ds^2 = \Lambda(x,y)(dx^2 + dy^2), \qquad H = \frac{p_1^2 + p_2^2}{2\Lambda}.$$

Theorem (V.V. Kozlov)
For any $n \geq 1, n \in \mathbb{N}$ there exists an analytic function $\Lambda(x, y)$ such that the corresponding Hamiltonian system possesses an irreducible polynomial integral of degree n with analytic (in a small neighborhood of a point $x = y = 0$) coefficients.

Polynomial integrals of geodesic flow

Let

$$F = a_n(x,y)p_1^n + a_{n-1}(x,y)p_1^{n-1}p_2 + \ldots + a_1(x,y)p_1p_2^{n-1} + a_0(x,y)p_2^n$$

be the first integral of this geodesic flow. The following relations hold:

$$\frac{\partial a_n}{\partial x}\Lambda + \frac{n}{2}a_n\frac{\partial \Lambda}{\partial x} + \frac{a_{n-1}}{2}\frac{\partial \Lambda}{\partial y} = 0,$$

$$\frac{\partial a_n}{\partial y}\Lambda + \frac{\partial a_{n-1}}{\partial x}\Lambda + \frac{n-1}{2}a_{n-1}\frac{\partial \Lambda}{\partial x} + a_{n-2}\frac{\partial \Lambda}{\partial y} = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{a_1}{2}\frac{\partial \Lambda}{\partial x} + \frac{\partial a_0}{\partial y}\Lambda + \frac{n}{2}a_0\frac{\partial \Lambda}{\partial y} = 0.$$

Let $a_1 \neq 0$. Then this system can be solved with respect to $\frac{\partial \Lambda}{\partial x}, \frac{\partial a_0}{\partial x}, \ldots, \frac{\partial a_n}{\partial x}$. So we can consider a Cauchy problem on the line $x = 0$ and apply the Cauchy–Kovalevskaya theorem to prove the existence and uniqueness of an analytic solution.

Classical hodograph method (n=2)

Consider a quasi-linear system of PDEs of the form

$$\begin{pmatrix} f \\ g \end{pmatrix}_y = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}_x, \qquad a_{ij} = a_{ij}(f,g)$$

on the unknown functions $f(x,y), g(x,y)$. The following relations hold:

$$\frac{\partial f}{\partial x} = \triangle \frac{\partial y}{\partial g},\ \frac{\partial f}{\partial y} = -\triangle \frac{\partial x}{\partial g},\ \frac{\partial g}{\partial x} = -\triangle \frac{\partial y}{\partial f},\ \frac{\partial g}{\partial y} = \triangle \frac{\partial x}{\partial f},$$

where $\triangle = \left(\frac{\partial x}{\partial f}\frac{\partial y}{\partial g} - \frac{\partial x}{\partial g}\frac{\partial y}{\partial f} \right)^{-1}$. We obtain the following system of linear PDEs:

$$-\frac{\partial x}{\partial g} = a_{11}(f,g)\frac{\partial y}{\partial g} - a_{12}(f,g)\frac{\partial y}{\partial f},$$

$$\frac{\partial x}{\partial f} = a_{21}(f,g)\frac{\partial y}{\partial g} - a_{22}(f,g)\frac{\partial y}{\partial f}.$$

Extended hodograph method (n=3)

Consider a quasi-linear system of PDEs of the form

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix}_y = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} f \\ g \\ h \end{pmatrix}_x, \qquad a_{ij} = a_{ij}(f,g,h)$$

on the unknown functions $f(x,y), g(x,y), h(x,y)$. To apply the hodograph method, we need an additional flow which commutes with the previous one:

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix}_t = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} f \\ g \\ h \end{pmatrix}_x, \qquad b_{ij} = b_{ij}(f,g,h).$$

Denote $\triangle = (t_f(x_h y_g - x_g y_h) - t_g(x_h y_f - x_f y_h) + t_h(x_g y_f - x_f y_g))^{-1}$. We have

$$\frac{\partial f}{\partial x} = \triangle \left(\frac{\partial y}{\partial h}\frac{\partial t}{\partial g} - \frac{\partial y}{\partial g}\frac{\partial t}{\partial h} \right), \ \frac{\partial f}{\partial y} = -\triangle \left(\frac{\partial x}{\partial h}\frac{\partial t}{\partial g} - \frac{\partial x}{\partial g}\frac{\partial t}{\partial h} \right), \ \frac{\partial f}{\partial t} = \triangle \left(\frac{\partial x}{\partial h}\frac{\partial y}{\partial g} - \frac{\partial x}{\partial g}\frac{\partial y}{\partial h} \right), \ldots$$

Semi-Hamiltonian systems, the generalized hodograph method

S.P. Tsarev.

A quasi-linear system of PDEs written in the diagonal form

$$r^i_t = v_i(r) r^i_x, \qquad i = 1, \ldots, n, \qquad v_i \neq v_j$$

is called *semi-Hamiltonian* if

$$\partial_{r_j} \frac{\partial_{r_i} v_k}{v_i - v_k} = \partial_{r_i} \frac{\partial_{r_j} v_k}{v_j - v_k}, \qquad i \neq j \neq k \neq i.$$

Here r_j are Riemann invariants. Semi-Hamiltonian systems possess infinitely many symmetries, i.e. commuting flows of the form $r^i_\tau = \omega_i(r) r^i_x$, $i = 1, \ldots, n$, wherein the following relations on v_i, ω_i hold:

$$\frac{\partial_{r_k} v_i}{v_k - v_i} = \frac{\partial_{r_k} \omega_i}{\omega_k - \omega_i}, \qquad i \neq k.$$

Due to the generalized hodograph method, a local solution is given by the following system of algebraic equations:

$$\omega_i(r) = v_i(r) t + x.$$

Semi-Hamiltonian systems, the generalized hodograph method

Consider a quasi-linear semi-Hamiltonian system of PDEs which is not in the diagonal form:

$$u_t^i = \sum_{j=1}^{n} v_j^i(u) u_x^j, \qquad i = 1, \dots, n.$$

Suppose that this system possesses a symmetry, i.e. a commuting flow of the form $u_\tau^i = \sum_{j=1}^n \omega_j^i(u) u_x^j,\ i = 1, \dots, n,$ wherein the mixed derivatives coincide:

$$\partial_\tau(u_t^i) = \partial_\tau \left(\sum_{j=1}^{n} v_j^i(u) u_x^j \right) = \partial_t(u_\tau^i) = \partial_t \left(\sum_{j=1}^{n} \omega_j^i(u) u_x^j \right).$$

Due to the generalized hodograph method, a local solution is given by the following system of algebraic equations:

$$x\delta_k^i + t v_k^i = \omega_k^i.$$

Polynomial integrals of the geodesic flow on a 2-surface

Theorem (G. Abdikalikova, A.E. Mironov)
On a 2-surface introduce the coordinates $ds^2 = g^2(t,x)dt^2 + dx^2$. *The Hamiltonian takes the form* $H = \frac{1}{2}\left(\frac{p_1^2}{g^2} + p_2^2\right)$. *The corresponding geodesic flow has a local polynomial in momenta first integral of the fourth degree:*

$$F_4 = \frac{a_0}{g^4}p_1^4 + \frac{a_1}{g^3}p_1^3p_2 + \frac{a_2}{g^2}p_1^2p_2^2 + p_1p_2^3 + p_2^4.$$

Here

$$a_0(t,x) = \frac{3(c_2 + t + 3c_3^2)}{5c_3^2}, \quad a_2(t,x) = -\frac{6(2c_2 + 2t + c_3^2)}{5c_3^2},$$

$$a_1(t,x) = -\frac{3\sqrt{c_3^2(-5c_1 - 4(3c_2 + 8t) - 18c_3^2 + 5x) - 12(c_2 + t)^2}}{5c_3^2},$$

$$g(t,x) = \frac{2\sqrt{c_3^2(-5c_1 - 4(3c_2 + 8t) - 18c_3^2 + 5x) - 12(c_2 + t)^2}}{5c_3^2},$$

where c_1, c_2, c_3 *are arbitrary constants.*

Rational integrals of geodesic flow

Choose the conformal coordinates (x, y), such that $H = \frac{p_1^2+p_2^2}{2\Lambda}$. Let U be a small neighborhood of a point $x = y = 0$. Denote P_r, Q_s — homogeneous in momenta p_1, p_2 polynomials of degrees r, s accordingly.

Theorem (V.V. Kozlov)
For any $r \geq 1, s \geq 1, r, s \in \mathbb{N}, r \geq s$ there exists an analytic function $\Lambda : U \to \mathbb{R}$ such that
1. the corresponding Hamiltonian system possesses an irreducible rational in momenta first integral (independent on the Hamiltonian) of the form

$$F = \frac{P_r}{Q_s}$$

with analytic coefficients in U;
2. polynomials P_r, Q_s are irreducible a.e. in U;
3. the Hamiltonian system doesn't possess any rational first integrals (independent on the Hamiltonian) of the form $F = \frac{P'_r}{Q'_s}$, $r' + s' < r + s$.

Rational integrals of the geodesic flow on a 2-surface

Maciejewski A.J., Przybylska M.: The following two functions commute

$$H = \frac{p_1^2 + p_2^2}{2} + f(p_1, p_2)(xp_1 - \alpha y p_2), \quad F = p_1^{\alpha} p_2, \quad \alpha \in \mathbb{R}.$$

- If $\alpha \in \mathbb{R}/\mathbb{Q}$, then F is not meromorphic.
- If $\alpha \in \mathbb{Q}$, then F is algebraic.
- If $\alpha \in -\mathbb{N}$, then F is rational.
- If $\alpha \in \mathbb{N}$, then F is polynomial.

Aoki A., Houri T., Tomoda K.: Let $f(p_1, p_2) = p_1 + p_2,\ \alpha = -\frac{s}{r}$ with relatively prime, positive integers r, s. Then

$$H = \left(x + \frac{1}{2}\right) p_1^2 + (x - \alpha y)\, p_1 p_2 + \left(\frac{1}{2} - \alpha y\right) p_2^2, \quad \widetilde{F} = F^r = \frac{p_2^r}{p_1^s}.$$

So we obtain a rational first integral $\widetilde{F}$ of the geodesic flow on a 2-surface (with the exceptional flat case $\alpha = -1$).

Rational integrals of the geodesic flow on a 2-surface

M.V. Pavlov, S.P. Tsarev

Suppose that the geodesic flow with the Hamiltonian

$$H = \frac{1}{2}\left(p_1^2 + \frac{p_2^2}{(b_1(x,t) - b_2(x,t))^2}\right)$$

possesses an additional first integral:

$$F = \frac{(b_1 - b_2)p_1 - b_2 p_2}{(b_1 - b_2)p_1 - b_1 p_2}.$$

Then the following relations hold:

$$b_{1t} = (1 + b_1 b_2) b_{1x} - (1 + b_1^2) b_{2x}, \quad b_{2t} = (1 + b_2^2) b_{1x} - (1 + b_1 b_2) b_{2x}.$$

This system turns out to be semi-Hamiltonian.

Rational integrals of the geodesic flow on a 2-surface

After making the following change of variables:

$$u(x,t) = -\frac{2}{(b_1 - b_2)^2}, \quad v(x,t) = \frac{b_1 + b_2}{b_1 - b_2}$$

we obtain the system of PDEs:

$$u_t + 2v_x = 0, \quad v_t + (\log u)_x = 0.$$

This system appears in fluid mechanics (barotropic fluid), gas dynamics (polytropic gas), also well known as a dispersionless limit of 2DToda lattice. In the hyperbolic domain this system can be written in the Riemann invariants:

$$r_{1x} = \frac{1}{4}(r_1 - r_2)\, r_{1t}, \quad r_{2x} = -\frac{1}{4}(r_1 - r_2)\, r_{2t},$$

where

$$u = -\frac{(r_1 - r_2)^2}{8}, \quad v = \frac{r_1 + r_2}{2}.$$

Rational integrals of the geodesic flow on a 2-surface

The hodograph method produces:

$$t_{r_2} = \frac{1}{4}\,(r_1 - r_2)\,x_{r_2}, \quad t_{r_1} = -\frac{1}{4}\,(r_1 - r_2)\,x_{r_1}.$$

By cross differentiation we obtain the Euler – Darboux – Poisson equation on $x(r_1, r_2)$:

$$\frac{\partial^2 x}{\partial r_1 \partial r_2} = \frac{1}{2(r_1 - r_2)}\left(\frac{\partial x}{\partial r_1} - \frac{\partial x}{\partial r_2}\right).$$

Having a solution of this equation, one may find t in quadratures.

Global rational integrals of natural mechanical systems

Consider a Hamiltonian system with Hamiltonian of the following form

$$H = \frac{p_1^2 + p_2^2}{2} + V(x, y),$$

where V is an analytic function on the plane $\mathbb{R}^2$ with the period lattice $\Lambda = \mathbb{Z}^2$.

Theorem (A.)
Suppose that this natural mechanical system possesses an additional global first integral of the form

$$F = \frac{a(x, y)p_1 + b(x, y)p_2 + c(x, y)}{f(x, y)p_1 + g(x, y)p_2 + h(x, y)}.$$

Then the potential has the form $V(x, y) = V_1(\alpha x + \beta y)$ and, consequently, there exists a linear in momenta first integral $F_1 = \alpha p_2 - \beta p_1$.

Thank you for your attention!