Redistribution of the combinatorial curvature under bistellar moves and local combinatorial formula for the first Pontryagin class

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(a joint work with Denis Gorodkov)

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Pontryagin classes of smooth manifolds

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(All three definitions are due to Pontryagin, 1940s).

Theorems of Rokhlin and Hirzebruch

Theorem (Rokhlin, 1952)

For an oriented smooth closed manifold M^4,

$$\mathrm{sign}(M^4) = \frac{1}{3} \left\langle p_1(M^4), [M^4] \right\rangle .$$

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Theorem (Hirzebruch, 1953)

For an oriented smooth closed manifold M^{4k},

$$\mathrm{sign}(M^{4k}) = \left\langle L_k\left(p_1(M^{4k}), \ldots, p_k(M^{4k})\right), [M^{4k}]\right\rangle.$$

Here

$$\prod_{i=1}^{\infty} \frac{\sqrt{t_i}}{\tanh \sqrt{t_i}} = 1 + \sum_{j=1}^{\infty} L_j(\sigma_1, \ldots, \sigma_j),$$

where σ_j is the jth elementary symmetric polynomial in t_i.

First Hirzebruch polynomials

$$\begin{aligned}
L_1(p_1) &= \frac{1}{3}p_1, \\
L_2(p_1,p_2) &= \frac{1}{45}(7p_2 - p_1^2), \\
L_3(p_1,p_2,p_3) &= \frac{1}{945}(62p_3 - 13p_2p_1 + 2p_1^3), \\
L_4(p_1,p_2,p_3,p_4) &= \frac{1}{14175}(381p_4 - 71p_3p_1 - 19p_2^2 + 22p_2p_1^2 - 3p_1^4), \\
&\dots
\end{aligned}$$

The Rokhlin–Schwarz–Thom construction

Theorem (Rokhlin–Schwarz, 1957; Thom, 1958)

The rational Pontryagin classes of smooth manifolds are invariant under PL homeomorphisms, and can be defined for PL manifolds without any smooth structure.

Idea of the proof: For any homology class $x \in H_{4k}(M^m; \mathbb{Z})$, there exists a positive integer k such that kx can be represented by a submanifold with trivial normal bundle:

$$N^{4k} \subset N^{4k} \times \Delta^{m+q-4k} \hookrightarrow M^m \times \mathbb{R}^q.$$

Define the Hirzebruch class $L_k(M^m) \in H^{4k}(M^m; \mathbb{Q})$ by

$$\langle L_k(M^m), x \rangle = \frac{\mathrm{sign}(N^{4k})}{k}$$

Now, the rational Pontryagin classes $p_i(M^m)$ can be written as polynomials in $L_k(M^m)$, $k = 1, \ldots, i$, with rational coefficients.

Topological invariance of rational Pontryagin classes

Theorem (Novikov, 1965)

The rational Pontryagin classes of manifolds are invariant under all homeomorphisms.

Combinatorial computation of rational Pontryagin classes

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Problem (Local computation of rational Pontryagin classes)

Given an (oriented) combinatorial simplicial manifold M^m, find an explicit formula for a cycle representing the Poincaré dual of $p_k(M^m) \in H^{4k}(M^m; \mathbb{Q})$ of the form

$$\sum_{\sigma^{m-4k} \in M^m} f(\operatorname{link} \sigma)\sigma,$$

where f is the $\mathbb{Q}$-valued function on the set of isomorphism classes of oriented $(4k-1)$-dimensional combinatorial simplicial spheres.

Link of a simplex

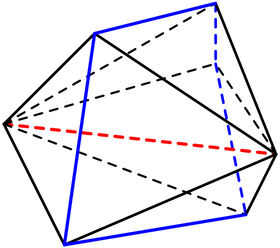
Let σ be a simplex of a simplicial complex K.

The star of σ is the subcomplex $\operatorname{star}\sigma$ of K consisting of all simplices containing σ and all their faces.

The link of σ is the subcomplex $\operatorname{link}\sigma$ of $\operatorname{star}\sigma$ consisting of all simplices disjoint from σ.

$$\operatorname{star}\sigma = \sigma * \operatorname{link}\sigma.$$

If K is a combinatorial manifold, then $\operatorname{link}\sigma$ is a combinatorial sphere of dimension $\dim K - \dim\sigma - 1$.



Algorithmic solution of the problem

G., 2008: For each k, presented an algorithm that

Given an oriented $(4k-1)$-dimensional combinatorial simplicial sphere L, computes a rational value $f(L)$ such that

for each combinatorial simplicial manifold M^m, the cycle

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represents the kth rational Pontryagin class of M^m.

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Unfortunately, this algorithm is extremely complicated (from the computational viewpoint) so that it cannot be realized even for simplest examples.

Gabrielov–Gelfand–Losik approach

Gabrielov–Gelfand–Losik, 1975:
A (more or less) explicit formula for the first rational Pontryagin class of a triangulated manifold with a given smoothing.

A given smoothing provides additional combinatorial data. Namely, at every vertex we get a fan consisting of cones tangent to simplices of the triangulation.

The approach was developed further and partially extended to higher Pontryagin classes by MacPherson (1977), Gabrielov (1978), Gelfand–MacPherson (1992).

None of the obtained formulae can be applied to a triangulated manifold without given smoothing.

Bistellar moves

Let K be an n-dimensional combinatorial simplicial manifold.

Suppose that K has a full subcomplex of the form $\sigma^k * \partial\tau^{n-k}$, where σ^k and τ^{n-k} are simplices.

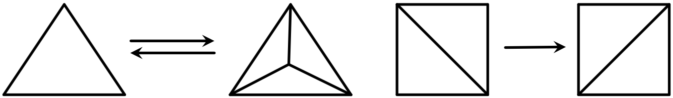
Replacing this subcomplex by $\partial\sigma^k * \tau^{n-k}$, we obtain a new combinatorial simplicial manifold that is PL homeomorphic to K.

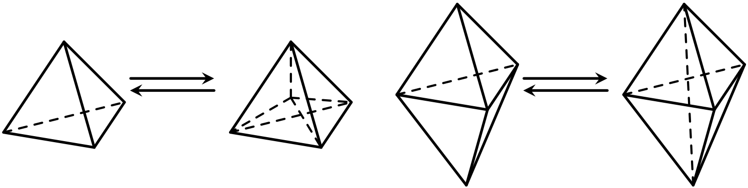
This operation is called a bistellar move.

Conventions: $\partial\sigma^0 = \varnothing, \quad \varnothing * \tau = \tau$.

Pachner, 1987:
If K_1 and K_2 are PL homeomorphic combinatorial simplicial manifolds, then K_1 can be transformed to K_2 by a sequence of bistellar moves.

Bistellar moves





Graph Γ_2 and cohomology class c

Graph Γ_2:
Vertices: Isomorphism classes of oriented simplicial 2-spheres.
Edges: Isomorphism classes of bistellar moves.

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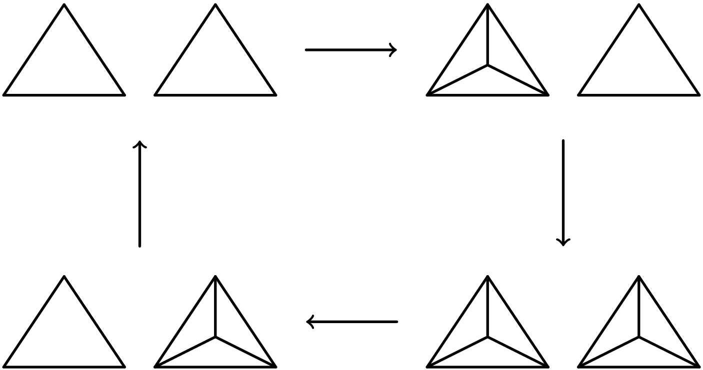
G.,2004:
Explicit formula for the first rational Pontryagin class via a special cohomology class $c \in H^1(\Gamma_2; \mathbb{Q})$.

To describe the cohomology class c, we shall write it values on certain elementary cycles that generate $H_1(\Gamma_2; \mathbb{Q})$.

Cohomology class c



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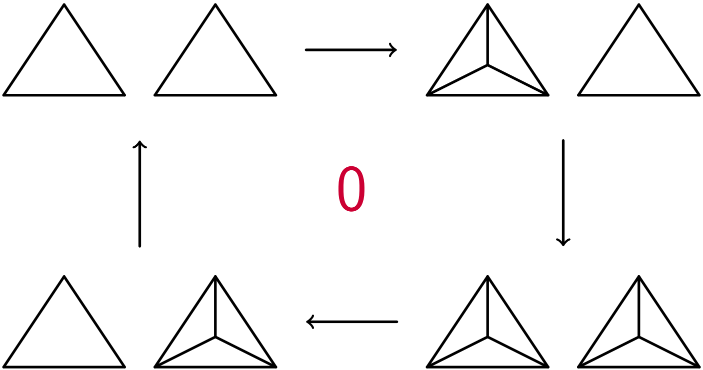

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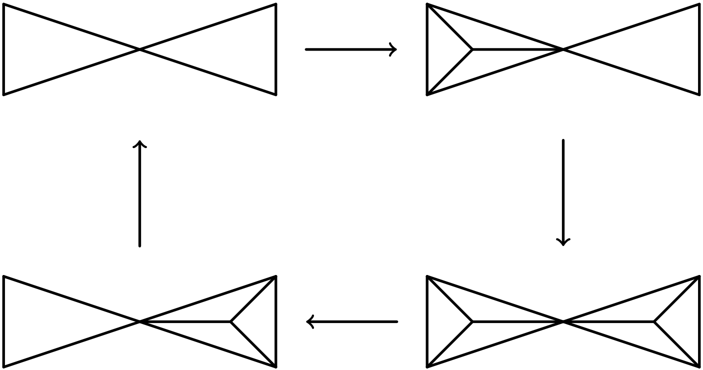

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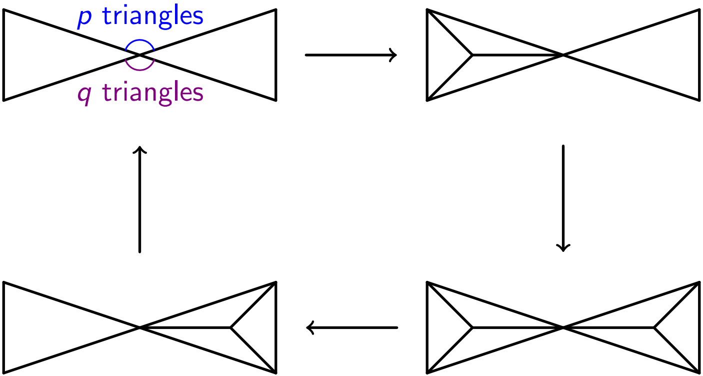

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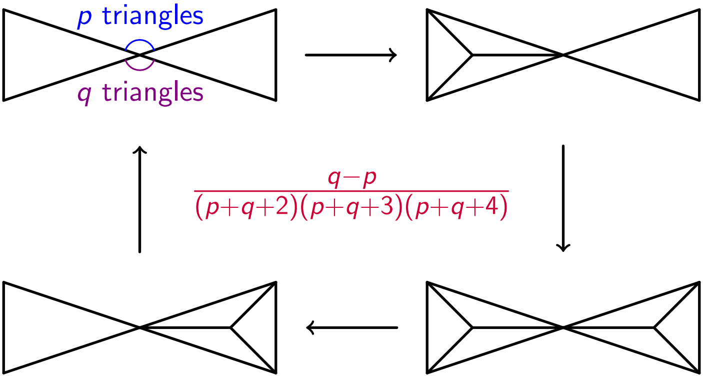
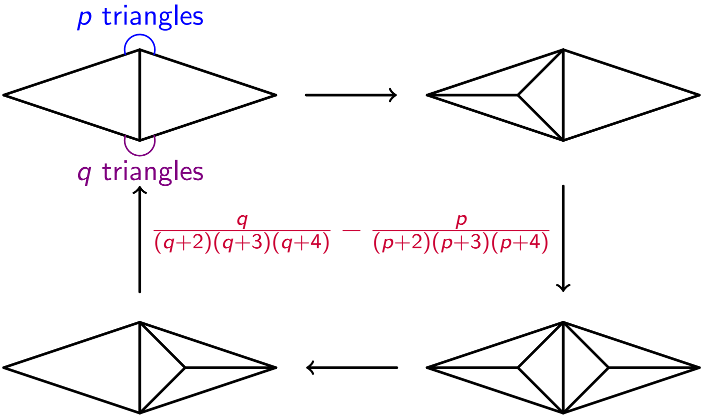


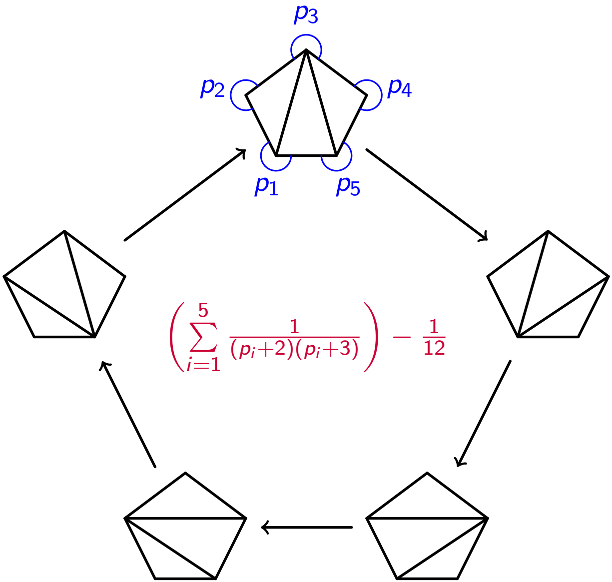
$$\frac{q-p}{(p+q+2)(p+q+3)(p+q+4)}$$

Cohomology class c



$$\frac{q}{(q+2)(q+3)(q+4)} - \frac{p}{(p+2)(p+3)(p+4)}$$

Cohomology class c



$$\left(\sum_{i=1}^{5} \frac{1}{(p_i+2)(p_i+3)}\right) - \frac{1}{12}$$

Formula

Theorem (G.,2004)

1. The cohomology class $c \in H^1(\Gamma_2;\mathbb{Q})$ *is well defined.*
2. Suppose that $h \in C^1(\Gamma_2;\mathbb{Q})$ *is a cocycle representing* c*. For an oriented simplicial 3-sphere* L*, take a sequence of bistellar moves*

$$\partial\Delta^4 \overset{\beta_1}{\rightsquigarrow} L_1 \overset{\beta_2}{\rightsquigarrow} L_2 \overset{\beta_3}{\rightsquigarrow} \cdots \overset{\beta_n}{\rightsquigarrow} L_n = L$$

and put

$$f(L) = \sum_{i=1}^{n} \sum_{v \text{ participates in } \beta_i} h(\beta_{i,v}),$$

where $\beta_{i,v}$ *is the bistellar move induced by* β_i *at the link of* v*.*
Then f *is a well-defined function on oriented simplicial 3-spheres.*
3. If M^m *is a combinatorial simplicial manifold, then the chain*

$$P_1 = \sum_{\sigma^{m-4}\in M^m} f(\operatorname{link}\sigma)\sigma \qquad \in \; C_{m-4}(M^m;\mathbb{Q})$$

is a cycle representing the Poincaré dual of $p_1(M^m)$*.*

Problem of finding a representative for c

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G.–Gorodkov, 2019:

An explicit formula for a cocycle h representing c.

Two main ingredients:

- Study of the redistribution of the combinatorial curvature under bistellar moves of 2-spheres.
- Generalized linking number of 1-cycles in a simplicial 3-sphere.

Combinatorial curvature at vertices

Let L be a two-dimensional simplicial sphere.

For a vertex $v \in L$, we denote by d_v the degree of v i. e. the number of edges entering v. The number

$$W_v = 1 - \frac{d_v}{6}$$

will be called the combinatorial curvature at v. Then

$$\sum_{v \in L} W_v = \chi(S^2) = 2.$$

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If we endow every triangle of L with the metric of a regular triangle with edge 1, then $2\pi W_v$ will be the integral curvature at v.

Combinatorial curvature at vertices: Probabilistic intuition

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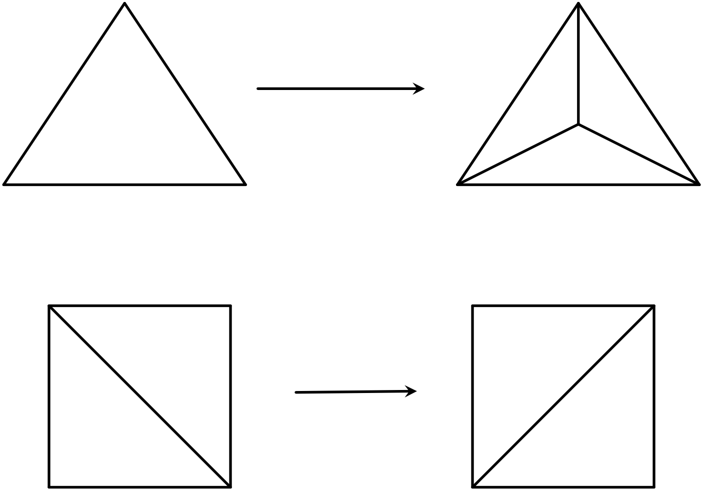
There exists no measure with such property. However, there exists a (unique) 'probabilistic' charge with this property:
The charge of every vertex v is $W_v/2$.

Redistribution of combinatorial curvature under moves



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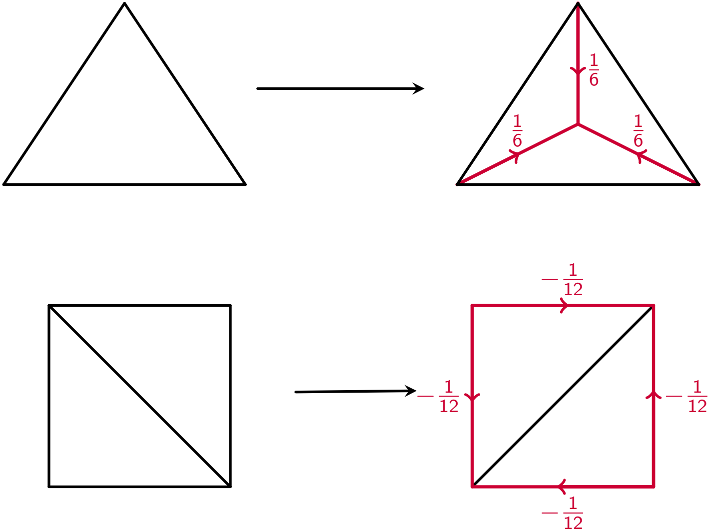
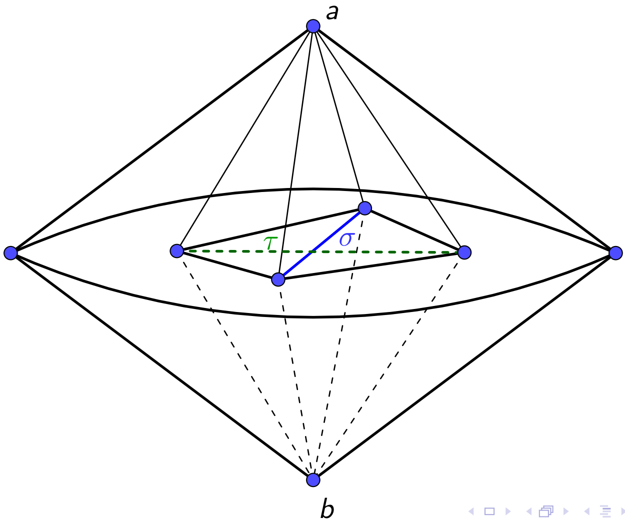


Three-dimensional simplicial sphere L_β

Let β be a bistellar move of simplicial 2-spheres transforming L_1 to L_2 and replacing $\sigma * \partial\tau$ with $\partial\sigma * \tau$. Then

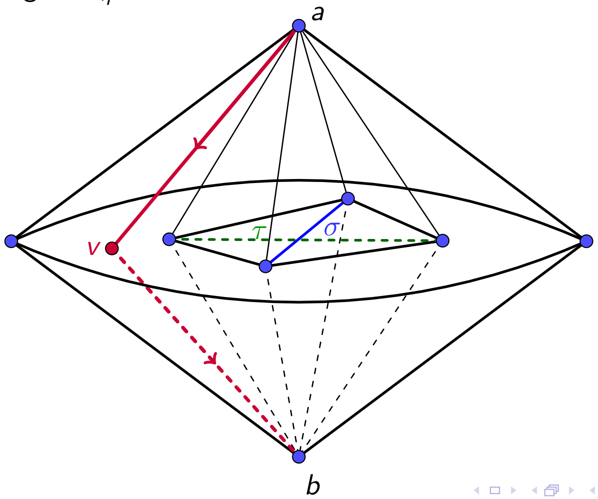
$$L_\beta = \mathrm{cone}(L_1) \cup \mathrm{cone}(L_2) \cup \{\sigma * \tau\}.$$



Set of chains $\mathcal{H}$

Consider the following finite set $\mathcal{H}$ of chains $\eta \in C_1(L_\beta; \mathbb{Z})$ with (rational) weights W_η assigned to them:

- For each vertex $v \notin \sigma \cup \tau$, take the chain $\eta = [av] + [vb]$ with weight $W_\eta = W_v$.



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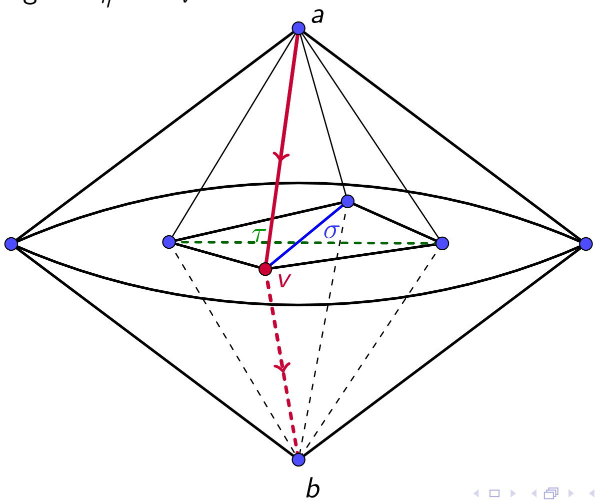
- For each vertex $v \in \sigma$, take the chain $\eta = [av] + [vb]$ with weight $W_\eta = W_v^{(L_2)}$.



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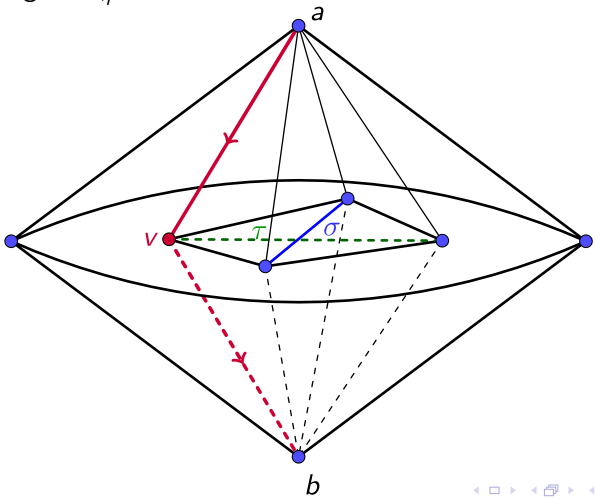
- For each vertex $v \in \tau$, take the chain $\eta = [av] + [vb]$ with weight $W_\eta = W_v^{(L_1)}$.



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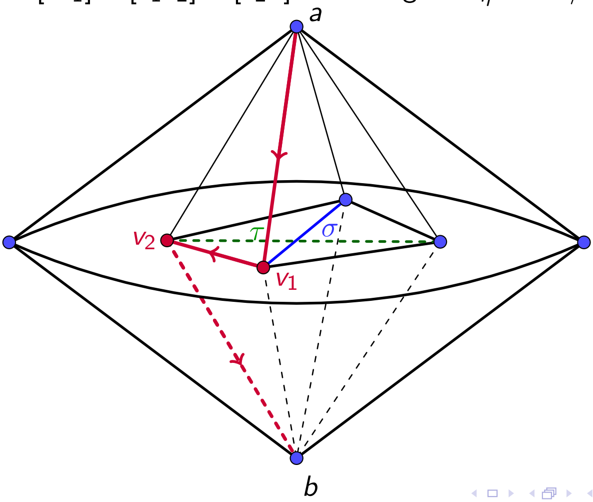
- For any two vertices $v_1 \in \sigma$, $v_2 \in \tau$, take the chain $\eta = [av_1] + [v_1 v_2] + [v_2 b]$ with weight $W_\eta = -1/12$ (or $1/6$).



The cycle $\xi_\beta \in Z_1(L_\beta;\mathbb{Q}) \otimes Z_1(L_\beta;\mathbb{Q})$

The set $\mathcal{H}$ of chains $\eta \in C_1(L_\beta;\mathbb{Z})$ with weights $W_\eta \in \mathbb{Q}$ satisfies:

- $\partial\eta = b - a$ for all $\eta \in \mathcal{H}$,
- $\sum_{\eta\in\mathcal{H}} W_\eta = \chi(S^2) = 2.$

Define the element $\xi_\beta \in C_1(L_\beta;\mathbb{Q}) \otimes C_1(L_\beta;\mathbb{Q})$ by

$$\xi_\beta = \sum_{\eta_1,\eta_2\in\mathcal{H}} W_{\eta_1} W_{\eta_2}\, \eta_1 \otimes \eta_2 - 2\sum_{\eta\in\mathcal{H}} W_\eta\, \eta \otimes \eta.$$

Proposition

The element ξ_β lies in the subgroup

$$Z_1(L_\beta;\mathbb{Q}) \otimes Z_1(L_\beta;\mathbb{Q}) \subset C_1(L_\beta;\mathbb{Q}) \otimes C_1(L_\beta;\mathbb{Q}).$$

Generalized linking number

Let L be an oriented 3-dimensional simplicial sphere, and let $\zeta_1, \zeta_2 \in Z_1(L;\mathbb{Q})$ be two cycles with disjoint supports. Then the linking number $\ell k(\zeta_1, \zeta_2)$ is well defined.

We would like to extend this definition to the case of arbitrary cycles $\zeta_1, \zeta_2 \in Z_1(L;\mathbb{Q})$.
To this end, we would like to shift ζ_2 off the 1-skeleton of L.

$$Shift\colon C_i(L;\mathbb{Q}) \to C_i(L^*;\mathbb{Q}), \qquad i = 0, 1,$$
$$Shift \circ \partial_L = \partial_{L^*} \circ Shift.$$

Now, put

$$\widetilde{\ell k}(\zeta_1, \zeta_2) = \ell k(\zeta_1, Shift(\zeta_2)),$$
$$\widetilde{\ell k}\colon Z_1(L;\mathbb{Q}) \otimes Z_1(L;\mathbb{Q}) \to \mathbb{Q}.$$

Shift operator

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1. For a vertex v, let $Shift(v)$ be the arithmetic mean of the barycentres of all tetrahedra containing v.
2. For an edge $[uv]$,

- let S_{uv} be the arithmetic mean of the barycentres of all tetrahedra containing both u and v,
- let $A \in C_1(Du;\mathbb{Q})$ be the chain of the smallest L^2-norm such that $\partial A = Shift(u) - S_{uv}$,
- let $B \in C_1(Dv;\mathbb{Q})$ be the chain of the smallest L^2-norm such that $\partial B = Shift(v) - S_{uv}$,
- put $Shift([uv]) = B - A$.

Cocycle $h \in C^1(\Gamma_2; \mathbb{Q})$

Let β be a bistellar move of oriented 2-dimensional simplicial spheres. Two ingredients:

$$\xi_\beta \in Z_1(L_\beta; \mathbb{Q}) \otimes Z_1(L_\beta; \mathbb{Q}),$$
$$\widetilde{\ell k} \colon Z_1(L_\beta; \mathbb{Q}) \otimes Z_1(L_\beta; \mathbb{Q}) \to \mathbb{Q}.$$

We put

$$h(\beta) = \widetilde{\ell k}(\xi_\beta).$$

Theorem (G.–Gorodkov, 2019)

The constructed cocycle $h \in C^1(\Gamma_2; \mathbb{Q})$ represents the cohomology class c described above.

The formula

For an oriented simplicial 3-sphere L, take a sequence of bistellar moves

$$\partial\Delta^4 \overset{\beta_1}{\rightsquigarrow} L_1 \overset{\beta_2}{\rightsquigarrow} L_2 \overset{\beta_3}{\rightsquigarrow} \cdots \overset{\beta_n}{\rightsquigarrow} L_n = L$$

and put

$$f(L) = \sum_{i=1}^{n} \sum_{v \text{ participates in } \beta_i} h(\beta_{i,v}),$$

where $\beta_{i,v}$ is the bistellar move induced by β_i at the link of v. Then f is a well-defined function on oriented simplicial 3-spheres.

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is a cycle representing the Poincaré dual of $p_1(M^m)$.

Thank you for your attention!