Distinguishing Legendrian and transverse links

Ivan Dynnikov

Steklov Mathematical Institute of Russian Academy of Sciences

December Readings — 2019

TSU, Tomsk, December 12, 2019

The talk is based on joint works (some of which are in progress) with

- Maxim Prasolov and
- Vladimir Shastin.

Oriented contact structure on a compact oriented M^3:

$$\xi = \ker_{\mathrm{or}} \alpha, \quad (\alpha \wedge d\alpha)_x \neq 0 \quad \forall x \in M^3.$$

Oriented contact structure on a compact oriented M^3:

$$\xi = \ker_{\mathrm{or}} \alpha, \quad (\alpha \wedge d\alpha)_x \neq 0 \quad \forall x \in M^3.$$

Equivalently: a 1-form α up to $\alpha \mapsto \lambda\alpha$, $\lambda(x) > 0\ \forall x$.

Oriented contact structure on a compact oriented M^3:

$$\xi = \ker_{\mathrm{or}} \alpha, \quad (\alpha \wedge d\alpha)_x \neq 0 \quad \forall x \in M^3.$$

Equivalently: a 1-form α up to $\alpha \mapsto \lambda\alpha$, $\lambda(x) > 0\ \forall x$.

A parametrized curve γ is ξ-*Legendrian* (respectively, ξ-*positively transverse*, ξ-*negatively transverse*), where $\xi = \ker_{\mathrm{or}} \alpha$, if

$$\alpha_{\gamma(t)}(\dot\gamma(t)) = 0 \quad \forall t$$

(respectively, $\alpha_{\gamma(t)}(\dot\gamma(t)) > 0$, $\alpha_{\gamma(t)}(\dot\gamma(t)) < 0$).

Oriented contact structure on a compact oriented M^3:

$$\xi = \ker_{\mathrm{or}} \alpha, \quad (\alpha \wedge d\alpha)_x \neq 0 \quad \forall x \in M^3.$$

Equivalently: a 1-form α up to $\alpha \mapsto \lambda\alpha$, $\lambda(x) > 0\ \forall x$.

A parametrized curve γ is ξ-*Legendrian* (respectively, ξ-*positively transverse*, ξ-*negatively transverse*), where $\xi = \ker_{\mathrm{or}} \alpha$, if

$$\alpha_{\gamma(t)}(\dot{\gamma}(t)) = 0 \quad \forall t$$

(respectively, $\alpha_{\gamma(t)}(\dot{\gamma}(t)) > 0$, $\alpha_{\gamma(t)}(\dot{\gamma}(t)) < 0$).

The standard contact structure in $\mathbb{R}^3$ (the default choice):

$$\xi_+ = \ker_{\mathrm{or}} \alpha_+, \text{ where } \alpha_+ = x\,dy + dz.$$

Oriented contact structure on a compact oriented M^3:

$$\xi = \ker_{\text{or}} \alpha, \quad (\alpha \wedge d\alpha)_x \neq 0 \quad \forall x \in M^3.$$

Equivalently: a 1-form α up to $\alpha \mapsto \lambda\alpha$, $\lambda(x) > 0\ \forall x$.

A parametrized curve γ is ξ*-Legendrian* (respectively, ξ*-positively transverse*, ξ*-negatively transverse*), where $\xi = \ker_{\text{or}} \alpha$, if

$$\alpha_{\gamma(t)}(\dot\gamma(t)) = 0 \quad \forall t$$

(respectively, $\alpha_{\gamma(t)}(\dot\gamma(t)) > 0$, $\alpha_{\gamma(t)}(\dot\gamma(t)) < 0$).

The standard contact structure in $\mathbb{R}^3$ (the default choice):

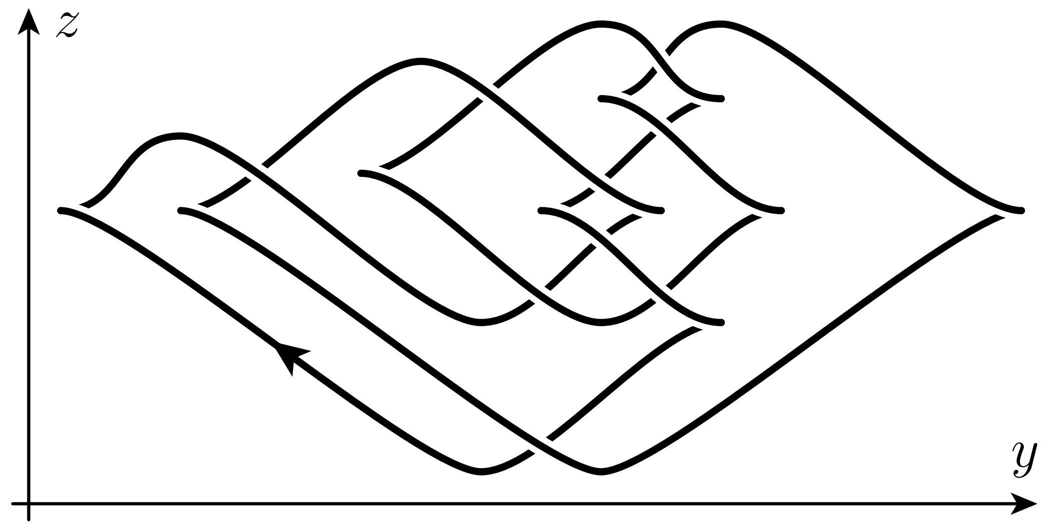
$$\xi_+ = \ker_{\text{or}} \alpha_+, \text{ where } \alpha_+ = x\, dy + dz.$$

We will also deal with the following mirror image of ξ_+:

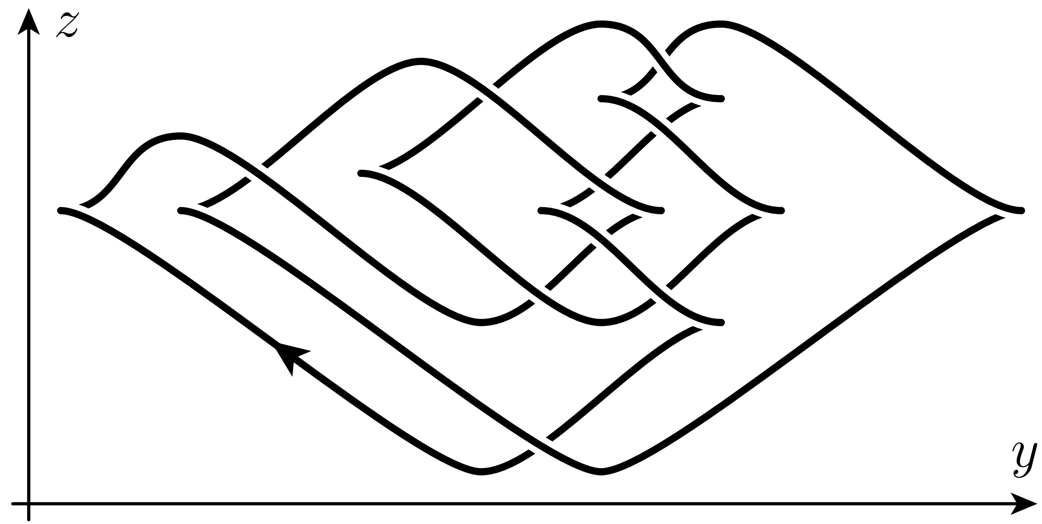
$$\xi_- = \ker_{\text{or}} \alpha_-, \text{ where } \alpha_- = -x\, dy + dz.$$

A (generic) Legendrian link is completely determined by its *front projection*, which is the projection to the yz-plane along the x-axis.

A (generic) Legendrian link is completely determined by its *front projection*, which is the projection to the yz-plane along the x-axis.



A (generic) Legendrian link is completely determined by its *front projection*, which is the projection to the yz-plane along the x-axis.

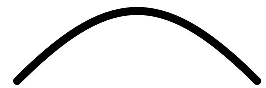


Along the curve: $x = -\frac{dz}{dy}$.

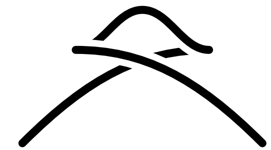
Reidemeister moves for Legendrian links

Reidemeister moves for Legendrian links

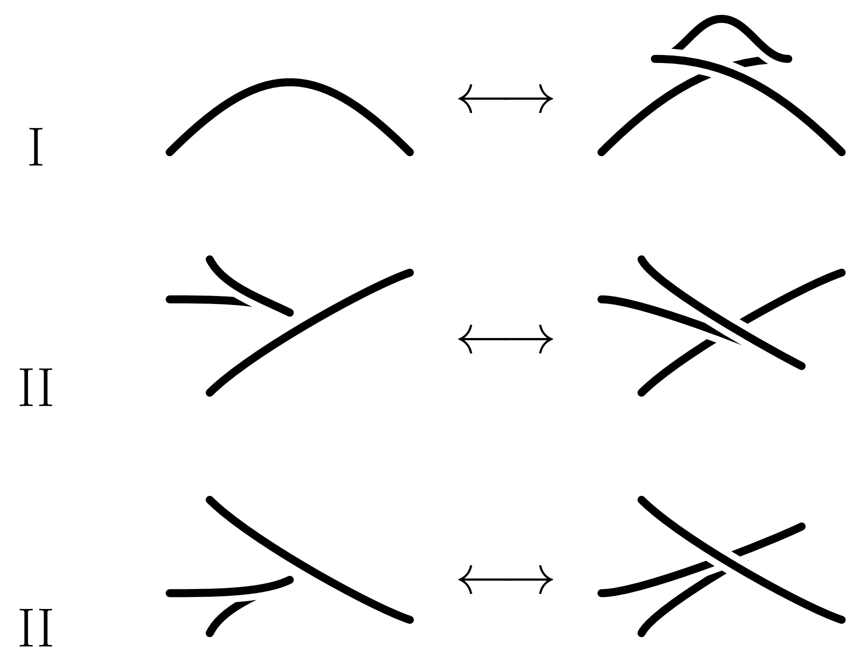
I



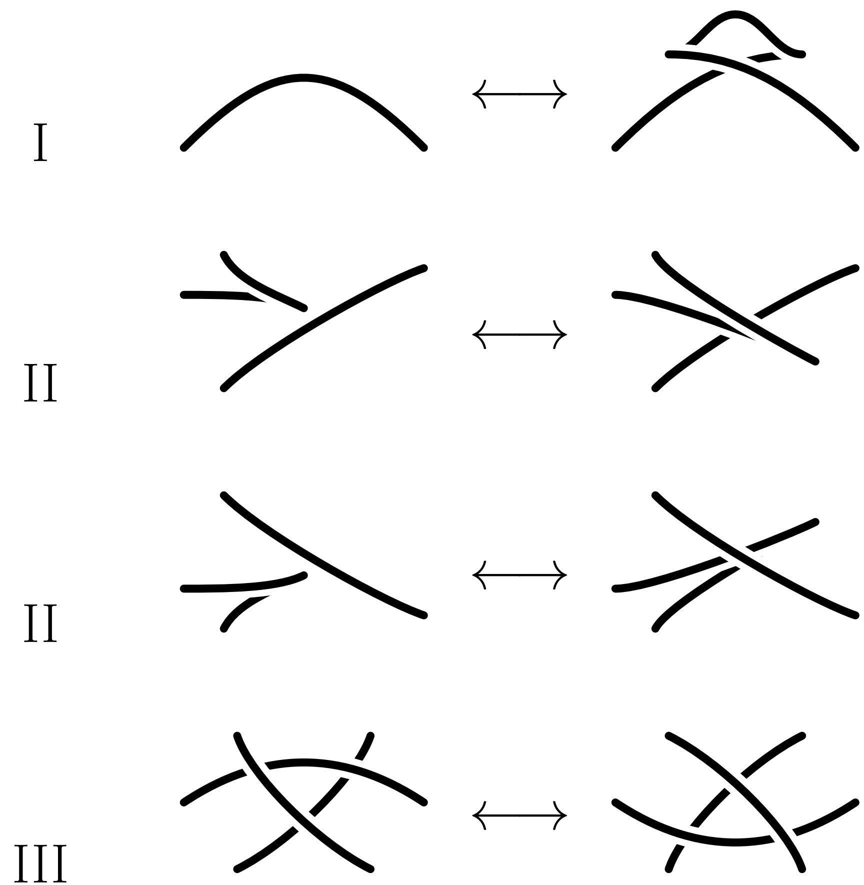
$\longleftrightarrow$



Reidemeister moves for Legendrian links

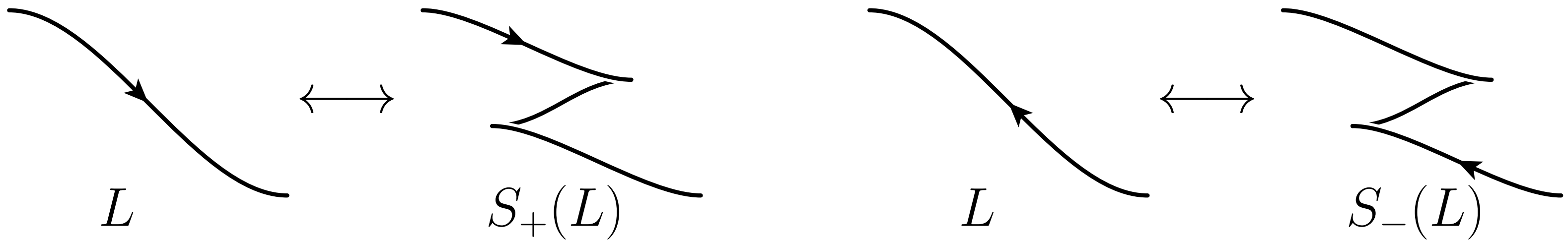


Reidemeister moves for Legendrian links



Stabilizations and destabilizations of Legendrian links

Stabilizations and destabilizations of Legendrian links



Stabilizations and destabilizations of Legendrian links

$$\frac{\{\text{Legendrian links}\}}{\text{Legendrian isotopy}, S_-} = \frac{\{\text{positively transverse links}\}}{\text{transverse isotopy}}$$

$$\frac{\{\text{Legendrian links}\}}{\text{Legendrian isotopy}, S_+} = \frac{\{\text{negatively transverse links}\}}{\text{transverse isotopy}}$$

Stabilizations and destabilizations of Legendrian links

$$\frac{\{\text{Legendrian links}\}}{\text{Legendrian isotopy, } S_-} = \frac{\{\text{positively transverse links}\}}{\text{transverse isotopy}}$$

$$\frac{\{\text{Legendrian links}\}}{\text{Legendrian isotopy, } S_+} = \frac{\{\text{negatively transverse links}\}}{\text{transverse isotopy}}$$

Two topologically equivalent Legendrian links became equivalent after some number of stabilizations (D.Fuchs–S.Tabachnikov, 1997).

Classical invariants of Legendrian links

Classical invariants of Legendrian links

Thurston–Bennequin number $\mathrm{tb}(K)$ of a Legendrian link K is defined as

$$\mathrm{tb}(K) = w - c/2,$$

where w is the writhe and c is the number of cups of the front projection of K.

Classical invariants of Legendrian links

Thurston–Bennequin number $\mathrm{tb}(K)$ of a Legendrian link K is defined as

$$\mathrm{tb}(K) = w - c/2,$$

where w is the writhe and c is the number of cups of the front projection of K.

Rotation number $r(K)$ of an oriented Legendrian link K is

$$\frac{1}{2}(c_- - c_+),$$

where c_+ (respectively, c_-) is the number of cusps oriented down (repsectively, oriented up).

Ya.Eliashberg, M.Fraser, 1995: Legendrian unknots having the same classical invariants are equivalent.

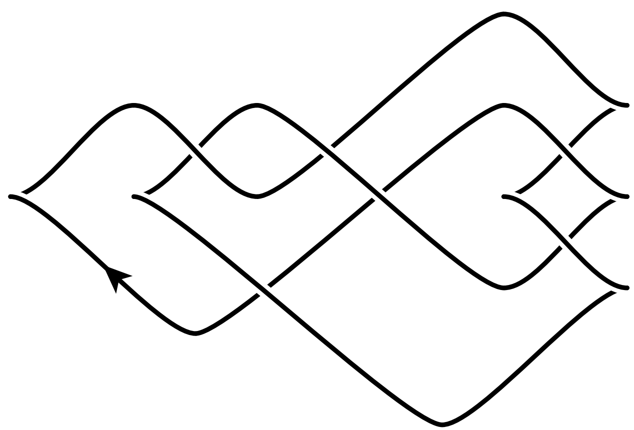
Ya.Eliashberg, M.Fraser, 1995: Legendrian unknots having the same classical invariants are equivalent.

Yu.Chekanov 1997: not true for non-trivial knots.

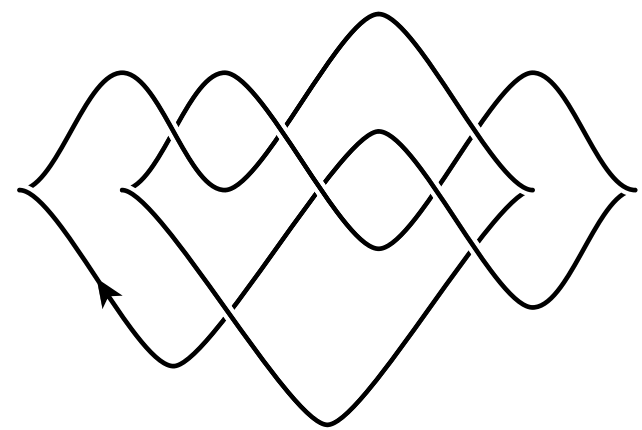
Ya.Eliashberg, M.Fraser, 1995: Legendrian unknots having the same classical invariants are equivalent.

Yu.Chekanov 1997: not true for non-trivial knots.

Example (knot type 5_2):



$\not\sim$



More Legendrian and transverse link invariants

- Ya. Eliashberg. Invariants in contact topology. *Doc. Math.* 1998, Extra Vol. II, 327–338.
- D. Fuchs. Chekanov–Eliashberg invariant of Legendrian knots: existence of augmentations. *J. Geom. Phys.* **47** (2003), no. 1, 43–65.
- L. Ng. Computable Legendrian Invariants. *Topology* **42** (2003), no. 1, 55–82.
- L. Ng. Combinatorial Knot Contact Homology and Transverse Knots. *Adv. Math.* **227** (2011), no. 6, 2189–2219.
- P. Pushkar', Yu. Chekanov. Combinatorics of fronts of Legendrian links and the Arnol'd 4-conjectures. *Russian Math. Surveys* **60** (2005), no. 1, 95–149.
- P. Ozsváth, Z. Szabó, D. Thurston. Legendrian Knots, Transverse Knots and Combinatorial Floer Homology. *Geom. Topol.* **12** (2008), no. 2, 941–980.

THE LEGENDRIAN KNOT ATLAS

WUTICHAI CHONGCHITMATE AND LENHARD NG

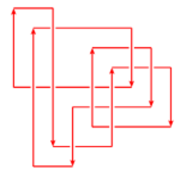
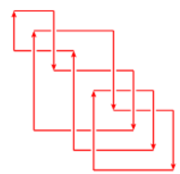
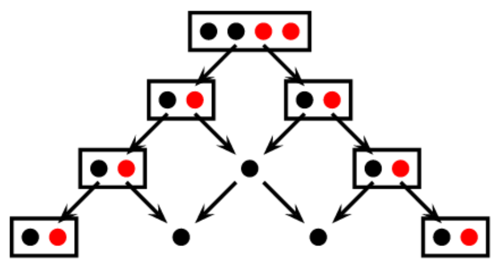
This is the Legendrian knot atlas, available online at

`http://www.math.duke.edu/~ng/atlas/`

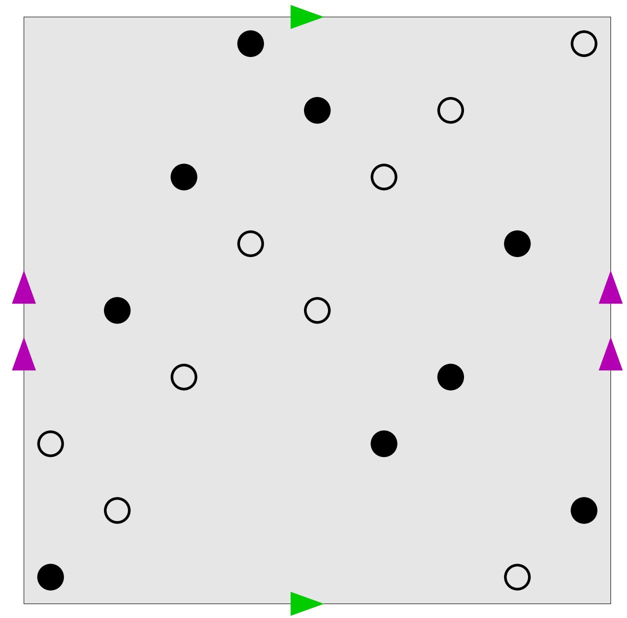
(permanent address: `http://alum.mit.edu/www/ng/atlas/`), and intended to accompany the paper "An atlas of Legendrian knots" by the authors [2]. This file was last changed on 22 October 2015.

The table on the following pages depicts conjectural classifications of Legendrian knots in all prime knot types of arc index up to 9. For each knot, we present a conjecturally complete list of non-destabilizable Legendrian representatives, modulo the symmetries of orientation reversal $L \mapsto -L$ and Legendrian mirroring $L \mapsto \mu(L)$. As usual, rotate 45° counterclockwise to translate from grid diagrams to fronts.

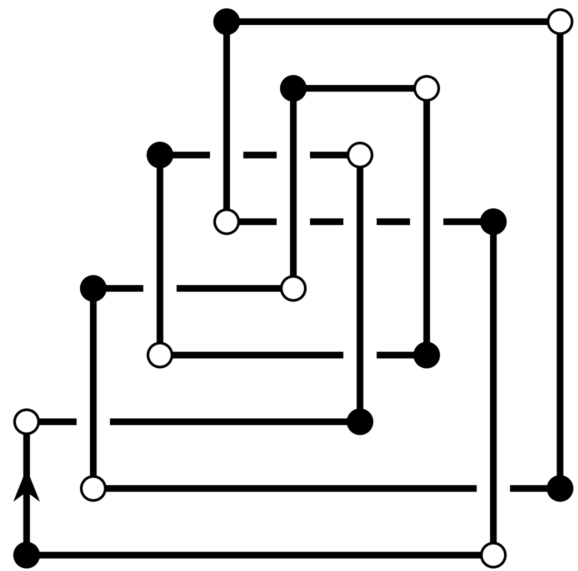
Each knot also comes with its conjectural Legendrian mountain range (extending infinitely downwards), comprised of black and red dots, plotted according to their Thurston–Bennequin number (vertical) and rotation

Knot Type	Grid Diagram	(tb, r)	$L = -L$?	$L = \mu(L)$?	$L = -\mu(L)$?
10_{128}		$(5, 0)$	✓	✗?	✗?
		$(5, 0)$	✗?	✓	✗?
					

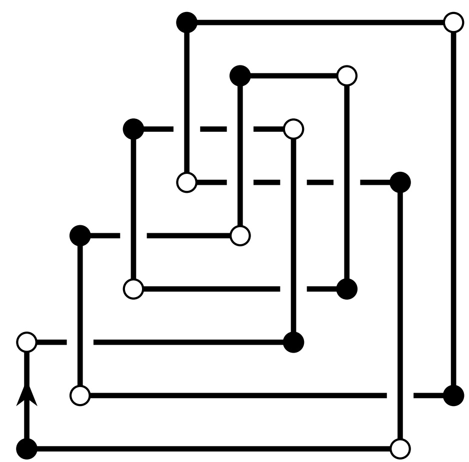
Oriented rectangular diagram of a link



Oriented rectangular diagram of a link

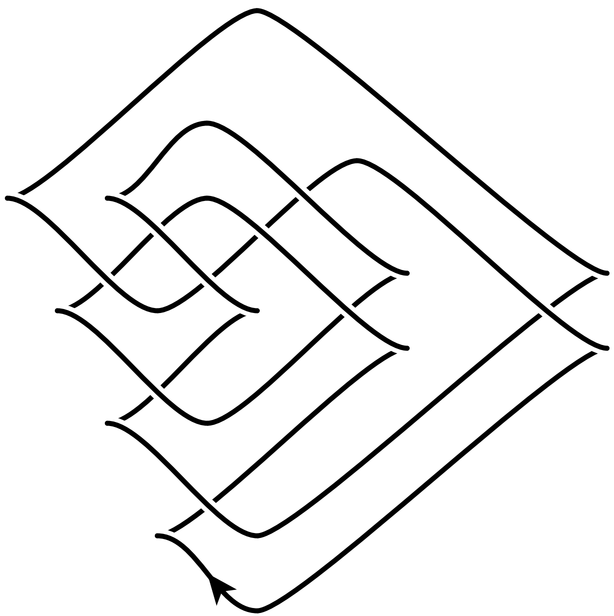


Rectangular diagrams $\longrightarrow$ Legendrian links

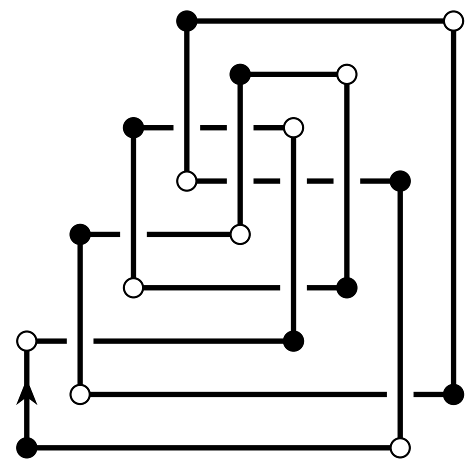


R

Rectangular diagrams $\longrightarrow$ Legendrian links

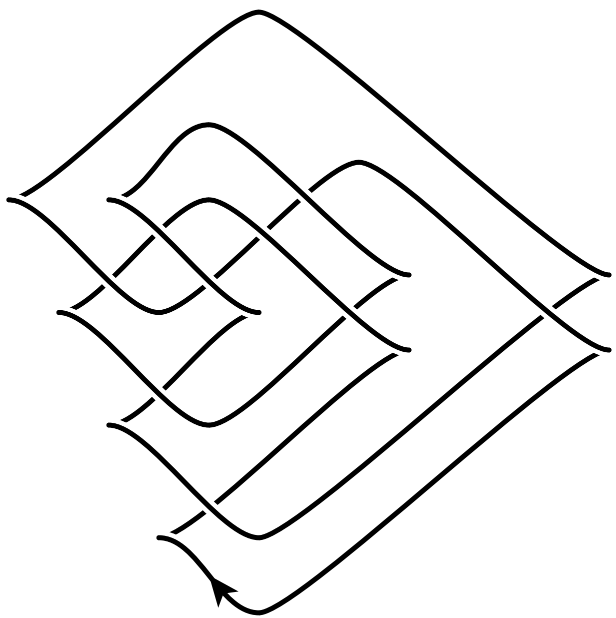


$L_+(R)$

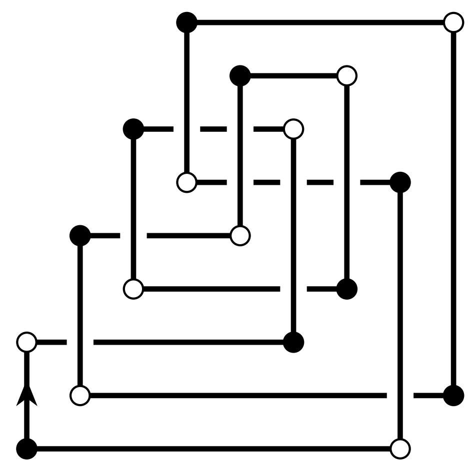


R

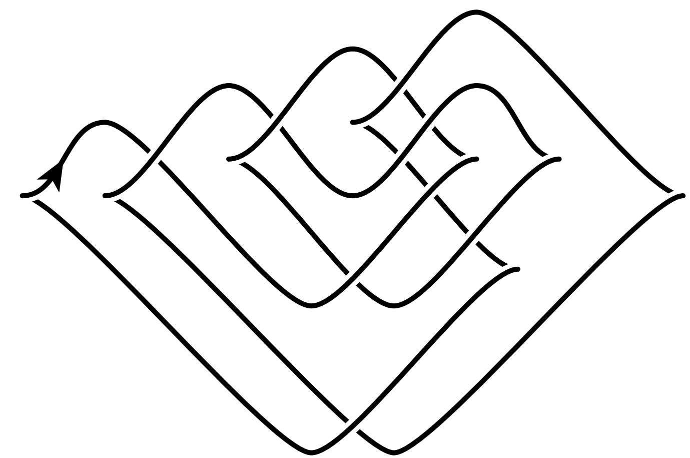
Rectangular diagrams $\longrightarrow$ Legendrian links



$L_+(R)$



R



$L_-(R)$

‘Reidemeister moves’ for rectangular diagrams:

‘Reidemeister moves’ for rectangular diagrams:

- exchange moves;

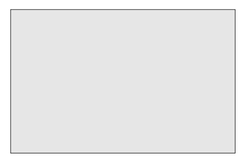
‘Reidemeister moves’ for rectangular diagrams:

- exchange moves;
- stabilizations and destabilizations.

‘Reidemeister moves’ for rectangular diagrams:

- exchange moves;
- stabilizations and destabilizations.

Conventions:

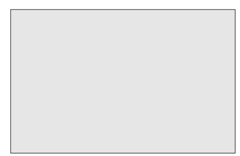


no vertices of the diagram except at the corners

‘Reidemeister moves’ for rectangular diagrams:

- exchange moves;
- stabilizations and destabilizations.

Conventions:



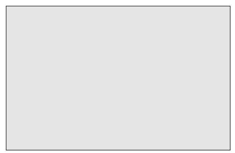
no vertices of the diagram except at the corners

vertices to be removed

‘Reidemeister moves’ for rectangular diagrams:

- exchange moves;
- stabilizations and destabilizations.

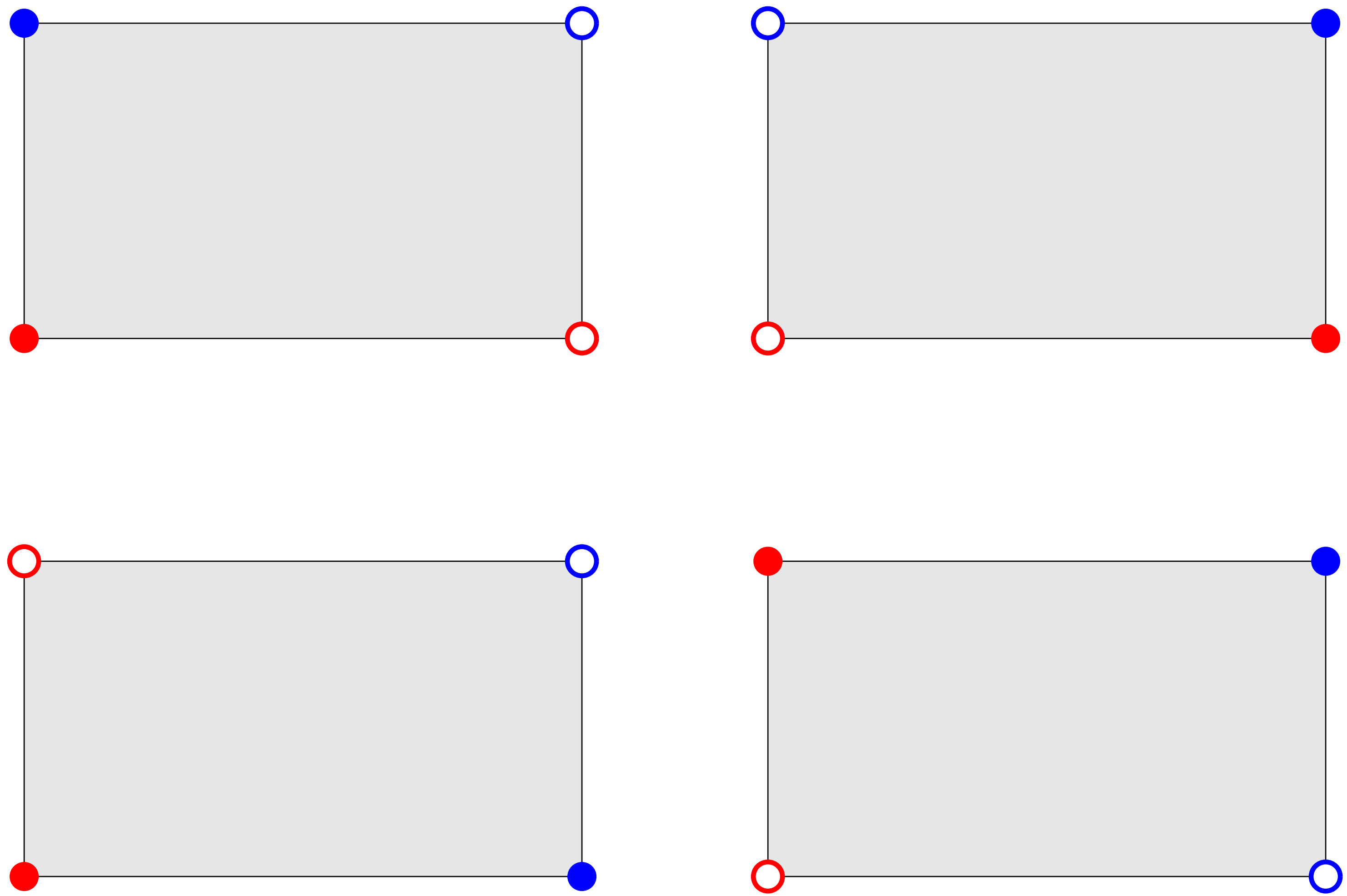
Conventions:


no vertices of the diagram except at the corners

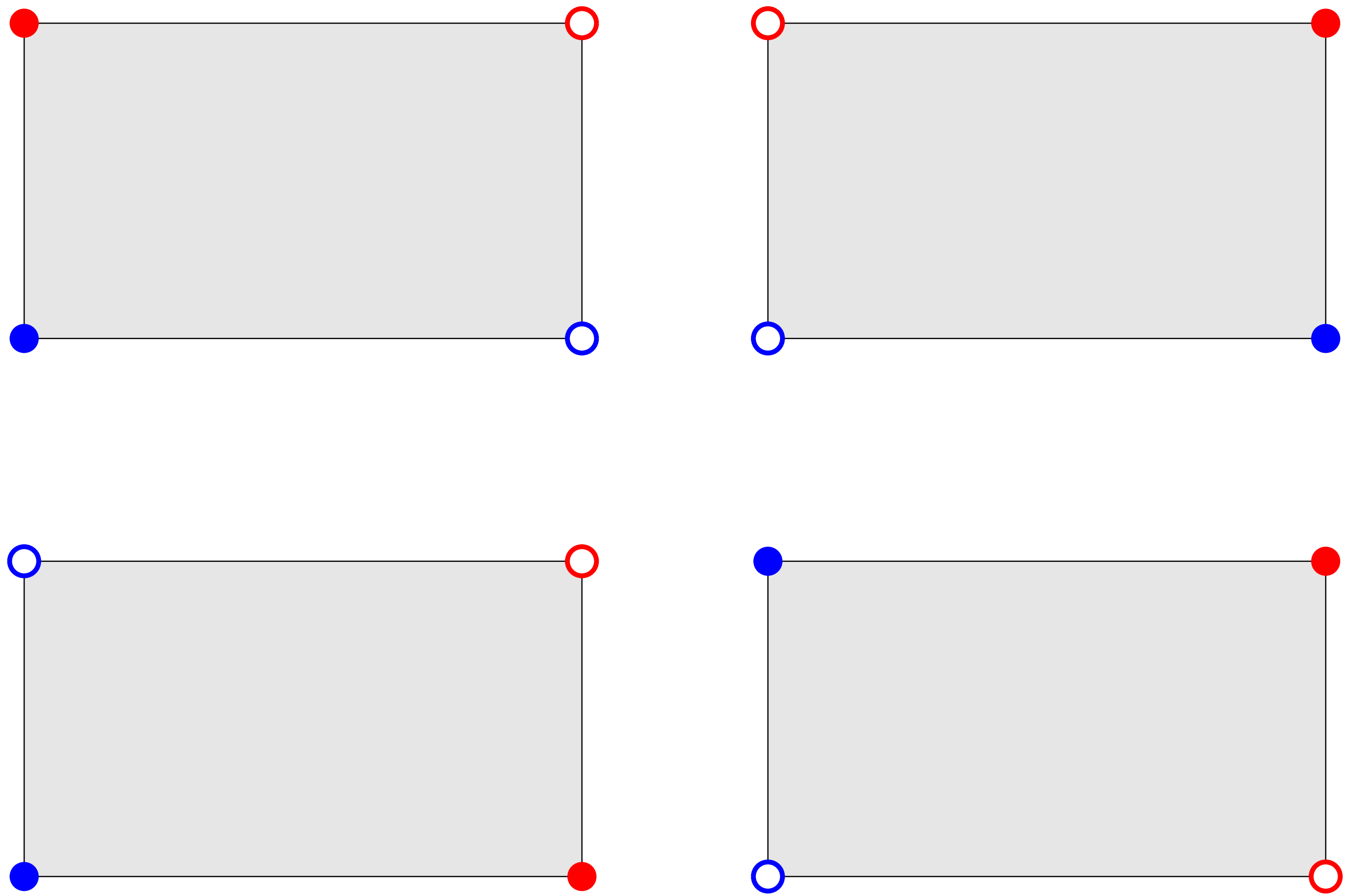

vertices to be removed


vertices to be added

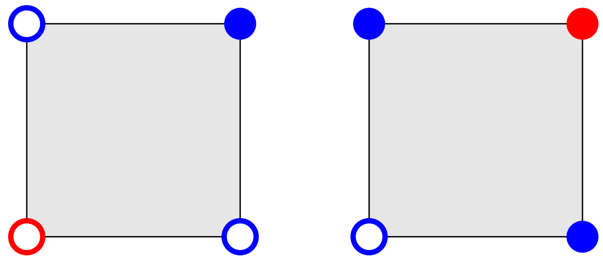
Exchange moves



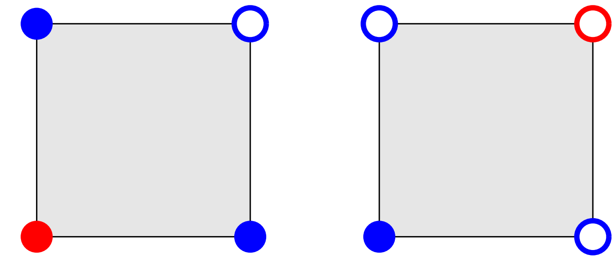
Exchange moves



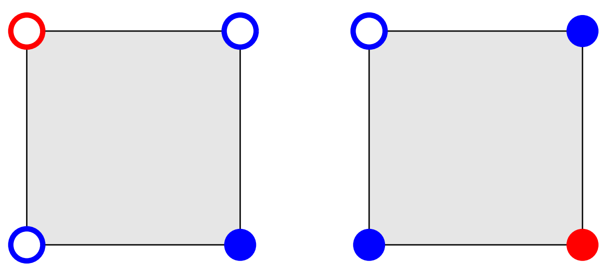
Stabilizations



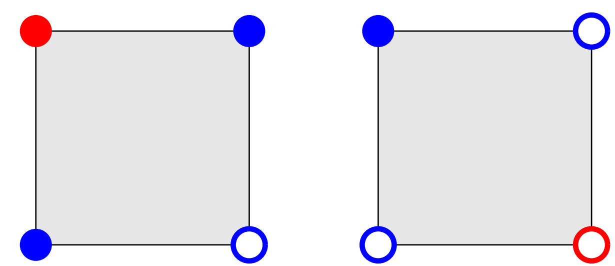
Type $\overrightarrow{\mathrm{I}}$



Type $\overleftarrow{\mathrm{I}}$

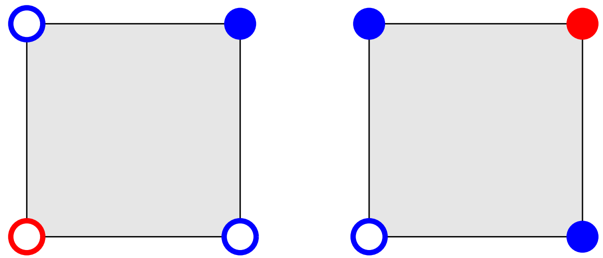


Type $\overrightarrow{\mathrm{II}}$

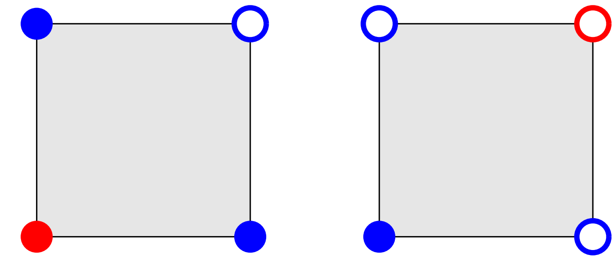


Type $\overleftarrow{\mathrm{II}}$

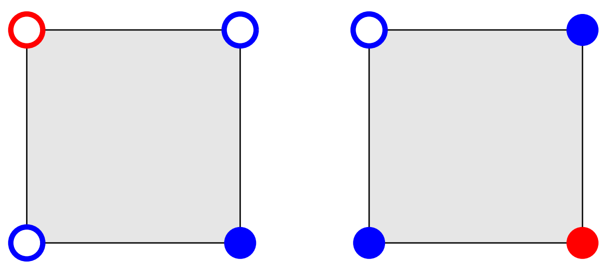
Stabilizations



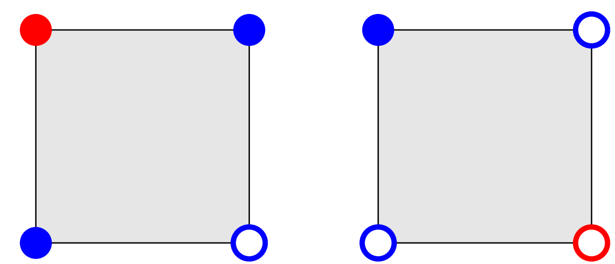
Type $\overrightarrow{\mathrm{I}}$



Type $\overleftarrow{\mathrm{I}}$



Type $\overrightarrow{\mathrm{II}}$



Type $\overleftarrow{\mathrm{II}}$

A destabilization = the inverse of a stabilization

$$\frac{\{\text{rectangular diagrams}\}}{\text{exch. moves}, S_{\overrightarrow{\mathrm{I}}}, S_{\overleftarrow{\mathrm{I}}}, S_{\overrightarrow{\mathrm{II}}}, S_{\overleftarrow{\mathrm{II}}}} = \{\text{links up to isotopy}\}$$

$$\frac{\{\text{rectangular diagrams}\}}{\text{exch. moves}, S_{\overrightarrow{\mathrm{I}}}, S_{\overleftarrow{\mathrm{I}}}, S_{\overrightarrow{\mathrm{II}}}, S_{\overleftarrow{\mathrm{II}}}} = \{\text{links up to isotopy}\}$$

$$\frac{\{\text{rectangular diagrams}\}}{\text{exchange moves}, S_{\overrightarrow{\mathrm{I}}}, S_{\overleftarrow{\mathrm{I}}},} = \{\text{Legendrian links up to Legendrian isotopy}\}$$

$$\frac{\{\text{rectangular diagrams}\}}{\text{exch. moves, } S_{\overrightarrow{\mathrm{I}}}, S_{\overleftarrow{\mathrm{I}}}, S_{\overrightarrow{\mathrm{II}}}, S_{\overleftarrow{\mathrm{II}}}} = \{\text{links up to isotopy}\}$$

$$\frac{\{\text{rectangular diagrams}\}}{\text{exchange moves, } S_{\overrightarrow{\mathrm{I}}}, S_{\overleftarrow{\mathrm{I}}},} = \{\text{Legendrian links up to Legendrian isotopy}\}$$

$$\frac{\{\text{rectangular diagrams}\}}{\text{exchange moves, } S_{\overrightarrow{\mathrm{I}}}, S_{\overrightarrow{\mathrm{II}}}} = \{\text{Birman–Menasco classes of braids}\}$$

$$\frac{\{\text{rectangular diagrams}\}}{\text{exch. moves}, S_{\overrightarrow{\mathrm{I}}}, S_{\overleftarrow{\mathrm{I}}}, S_{\overrightarrow{\mathrm{II}}}, S_{\overleftarrow{\mathrm{II}}}} = \{\text{links up to isotopy}\}$$

$$\frac{\{\text{rectangular diagrams}\}}{\text{exchange moves}, S_{\overrightarrow{\mathrm{I}}}, S_{\overleftarrow{\mathrm{I}}},} = \{\text{Legendrian links up to Legendrian isotopy}\}$$

$$\frac{\{\text{rectangular diagrams}\}}{\text{exchange moves}, S_{\overrightarrow{\mathrm{I}}}, S_{\overrightarrow{\mathrm{II}}}} = \{\text{Birman–Menasco classes of braids}\}$$

$$\frac{\{\text{rectangular diagrams}\}}{\text{exchange moves}, S_{\overrightarrow{\mathrm{I}}}, S_{\overleftarrow{\mathrm{I}}}, S_{\overrightarrow{\mathrm{II}}}} = \{\text{transverse links up to transverse isotopy}\}$$

$$\frac{\{\text{rectangular diagrams}\}}{\text{exch. moves}, S_{\overrightarrow{\mathrm{I}}}, S_{\overleftarrow{\mathrm{I}}}, S_{\overrightarrow{\mathrm{II}}}, S_{\overleftarrow{\mathrm{II}}}} = \{\text{links up to isotopy}\}$$

$$\frac{\{\text{rectangular diagrams}\}}{\text{exchange moves}, S_{\overrightarrow{\mathrm{I}}}, S_{\overleftarrow{\mathrm{I}}},} = \{\text{Legendrian links up to Legendrian isotopy}\}$$

$$\frac{\{\text{rectangular diagrams}\}}{\text{exchange moves}, S_{\overrightarrow{\mathrm{I}}}, S_{\overrightarrow{\mathrm{II}}}} = \{\text{Birman–Menasco classes of braids}\}$$

$$\frac{\{\text{rectangular diagrams}\}}{\text{exchange moves}, S_{\overrightarrow{\mathrm{I}}}, S_{\overleftarrow{\mathrm{I}}}, S_{\overrightarrow{\mathrm{II}}}} = \{\text{transverse links up to transverse isotopy}\}$$

By $[R]_{T_1,\ldots,T_k}$ we denote the class of R in $\dfrac{\{\text{rectangular diagrams}\}}{\text{exchange moves}, S_{T_1}, \ldots, S_{T_k}}$.

$$\frac{\{\text{rectangular diagrams}\}}{\text{exch. moves}, S_{\overrightarrow{\mathrm{I}}}, S_{\overleftarrow{\mathrm{I}}}, S_{\overrightarrow{\mathrm{II}}}, S_{\overleftarrow{\mathrm{II}}}} = \{\text{links up to isotopy}\}$$

$$\frac{\{\text{rectangular diagrams}\}}{\text{exchange moves}, S_{\overrightarrow{\mathrm{I}}}, S_{\overleftarrow{\mathrm{I}}},} = \{\text{Legendrian links up to Legendrian isotopy}\}$$

$$\frac{\{\text{rectangular diagrams}\}}{\text{exchange moves}, S_{\overrightarrow{\mathrm{I}}}, S_{\overrightarrow{\mathrm{II}}}} = \{\text{Birman–Menasco classes of braids}\}$$

$$\frac{\{\text{rectangular diagrams}\}}{\text{exchange moves}, S_{\overrightarrow{\mathrm{I}}}, S_{\overleftarrow{\mathrm{I}}}, S_{\overrightarrow{\mathrm{II}}}} = \{\text{transverse links up to transverse isotopy}\}$$

By $[R]_{T_1,\ldots,T_k}$ we denote the class of R in $\dfrac{\{\text{rectangular diagrams}\}}{\text{exchange moves}, S_{T_1}, \ldots, S_{T_k}}$.

In particular, $[R]$ is the *exchange class* of R.

Stabilizations are well defined on exchange classes: if $R_1 \mapsto R_2$ is a stabilization, then, for any $R_1' \in [R_1]$, there is a stabilization $R_1' \mapsto R_2'$ of the same type such that $R_2' \in [R_2]$.

With every rectangular diagram of a link R we associate an actual smooth link $\widehat{R} \subset \mathbb{S}^3$.

With every rectangular diagram of a link R we associate an actual smooth link $\widehat{R} \subset \mathbb{S}^3$.

With every composable sequence of elementary moves $s : R \mapsto R'$ we associate an element $\widehat{s}$ of

$$\mathrm{Diff}_{++}\big((\mathbb{S}^3, \widehat{R}), (\mathbb{S}^3, \widehat{R'})\big)/ \sim$$

so that $\widehat{s_1 s_2} = \widehat{s}_2 \circ \widehat{s}_1$ holds for composable sequences.

With every rectangular diagram of a link R we associate an actual smooth link $\widehat{R} \subset \mathbb{S}^3$.

With every composable sequence of elementary moves $s : R \mapsto R'$ we associate an element $\widehat{s}$ of

$$\mathrm{Diff}_{++}\big((\mathbb{S}^3, \widehat{R}), (\mathbb{S}^3, \widehat{R'})\big)/\sim$$

so that $\widehat{s_1 s_2} = \widehat{s}_2 \circ \widehat{s}_1$ holds for composable sequences.

If $R = R'$, then $\widehat{s}$ is an element of the *orientation-preserving symmetry group* of $\widehat{R}$:

$$\widehat{s} \in \mathrm{Sym}(\widehat{R}) = \mathrm{Diff}_{++}\big((\mathbb{S}^3, \widehat{R}), (\mathbb{S}^3, \widehat{R})\big)/\sim .$$

Main technical results

Commutation theorem: for any composable sequence $s : R \mapsto R'$ of elementary moves producing R' from R there are composable sequences of elementary moves $s_{\mathrm{I}} : R \mapsto R''$ and $s_{\mathrm{II}} : R'' \mapsto R'$ such that:

1. $\widehat{s} = \widehat{s_{\mathrm{I}} s_{\mathrm{II}}}$;

2. s_{I} (respectively, s_{II}) includes only exchange moves and type I (respectively, type II) stabilizations and destabilizations.

Main technical results

Commutation theorem: for any composable sequence $s : R \mapsto R'$ of elementary moves producing R' from R there are composable sequences of elementary moves $s_{\mathrm{I}} : R \mapsto R''$ and $s_{\mathrm{II}} : R'' \mapsto R'$ such that:

1. $\widehat{s} = \widehat{s_{\mathrm{I}} s_{\mathrm{II}}}$;

2. s_{I} (respectively, s_{II}) includes only exchange moves and type I (respectively, type II) stabilizations and destabilizations.

Non-triviality theorem: if s, s_{I}, s_{II} are as above and $R = R'$, $[R] \neq [R'']$, then

$$\widehat{s} \neq 1 \in \mathrm{Sym}(\widehat{R}).$$

Main technical results

Commutation theorem: for any composable sequence $s : R \mapsto R'$ of elementary moves producing R' from R there are composable sequences of elementary moves $s_{\mathrm{I}} : R \mapsto R''$ and $s_{\mathrm{II}} : R'' \mapsto R'$ such that:

1. $\widehat{s} = \widehat{s_{\mathrm{I}} s_{\mathrm{II}}}$;
2. s_{I} (respectively, s_{II}) includes only exchange moves and type I (respectively, type II) stabilizations and destabilizations.

Non-triviality theorem: if s, s_{I}, s_{II} are as above and $R = R'$, $[R] \neq [R'']$, then

$$\widehat{s} \neq 1 \in \mathrm{Sym}(\widehat{R}).$$

Partial diamond lemma: If $[R_1] \mapsto [R_0]$ and $[R_2] \mapsto [R_0]$ are stabilizations of types T_1 and T_2, respectively, with $T_1 \in \{\overrightarrow{\mathrm{I}}, \overleftarrow{\mathrm{I}}\}$, $T_2 \in \{\overrightarrow{\mathrm{II}}, \overleftarrow{\mathrm{II}}\}$, then there exist a rectangular diagram R_3 and stabilizations $[R_3] \mapsto [R_1]$, $[R_3] \mapsto [R_2]$ of types T_2, T_1, respectivley.

$$\frac{\{\text{rectangular diagrams}\}}{\text{exch. moves}, S_{\overrightarrow{\mathrm{I}}}, S_{\overleftarrow{\mathrm{I}}}, S_{\overrightarrow{\mathrm{II}}}, S_{\overleftarrow{\mathrm{II}}}} = \{\text{links up to isotopy}\}$$

$$\frac{\{\text{rectangular diagrams}\}}{\text{exchange moves}, S_{\overrightarrow{\mathrm{I}}}, S_{\overleftarrow{\mathrm{I}}},} = \{\text{Legendrian links up to Legendrian isotopy}\}$$

$$\frac{\{\text{rectangular diagrams}\}}{\text{exchange moves}, S_{\overrightarrow{\mathrm{I}}}, S_{\overrightarrow{\mathrm{II}}}} = \{\text{Birman–Menasco classes of braids}\}$$

$$\frac{\{\text{rectangular diagrams}\}}{\text{exchange moves}, S_{\overrightarrow{\mathrm{I}}}, S_{\overleftarrow{\mathrm{I}}}, S_{\overrightarrow{\mathrm{II}}}} = \{\text{transverse links up to transverse isotopy}\}$$

How to decide whether or not $[R_1]_{\overrightarrow{\mathrm{I}},\overleftarrow{\mathrm{I}}} = [R_2]_{\overrightarrow{\mathrm{I}},\overleftarrow{\mathrm{I}}}$?

How to decide whether or not $[R_1]_{\overrightarrow{\mathrm{I}},\overleftarrow{\mathrm{I}}} = [R_2]_{\overrightarrow{\mathrm{I}},\overleftarrow{\mathrm{I}}}$?

We may assume $[R_1]_{\overrightarrow{\mathrm{II}},\overleftarrow{\mathrm{II}}} = [R_2]_{\overrightarrow{\mathrm{II}},\overleftarrow{\mathrm{II}}}$.

How to decide whether or not $[R_1]_{\overrightarrow{\mathrm{I}},\overleftarrow{\mathrm{I}}} = [R_2]_{\overrightarrow{\mathrm{I}},\overleftarrow{\mathrm{I}}}$?

We may assume $[R_1]_{\overrightarrow{\mathrm{II}},\overleftarrow{\mathrm{II}}} = [R_2]_{\overrightarrow{\mathrm{II}},\overleftarrow{\mathrm{II}}}$.

Then if $\mathrm{Sym}(\widehat{R}_1) = \{1\}$ and $[R_1] \neq [R_2]$, then $[R_1]_{\overrightarrow{\mathrm{I}},\overleftarrow{\mathrm{I}}} \neq [R_2]_{\overrightarrow{\mathrm{I}},\overleftarrow{\mathrm{I}}}$

How to decide whether or not $[R_1]_{\overrightarrow{\mathrm{I}},\overleftarrow{\mathrm{I}}} = [R_2]_{\overrightarrow{\mathrm{I}},\overleftarrow{\mathrm{I}}}$?

We may assume $[R_1]_{\overrightarrow{\mathrm{II}},\overleftarrow{\mathrm{II}}} = [R_2]_{\overrightarrow{\mathrm{II}},\overleftarrow{\mathrm{II}}}$.

Then if $\mathrm{Sym}(\widehat{R}_1) = \{1\}$ and $[R_1] \neq [R_2]$, then $[R_1]_{\overrightarrow{\mathrm{I}},\overleftarrow{\mathrm{I}}} \neq [R_2]_{\overrightarrow{\mathrm{I}},\overleftarrow{\mathrm{I}}}$

In general, there is an algorithm to find finitely many composable sequences of elementary moves $s_1, \ldots, s_m : R_1 \mapsto R_1$ such that $\widehat{s}_1, \ldots, \widehat{s}_m$ generate $\mathrm{Sym}(\widehat{R}_1)$. From them we learn how many type I stabilizations have to be applied to $[R_1]$ and $[R_2]$ to get the same exchange class provided that $[R_1]_{\overrightarrow{\mathrm{I}},\overleftarrow{\mathrm{I}}} = [R_2]_{\overrightarrow{\mathrm{I}},\overleftarrow{\mathrm{I}}}$.

How to decide whether or not $[R_1]_{\overrightarrow{\mathrm{I}},\overleftarrow{\mathrm{I}}} = [R_2]_{\overrightarrow{\mathrm{I}},\overleftarrow{\mathrm{I}}}$?

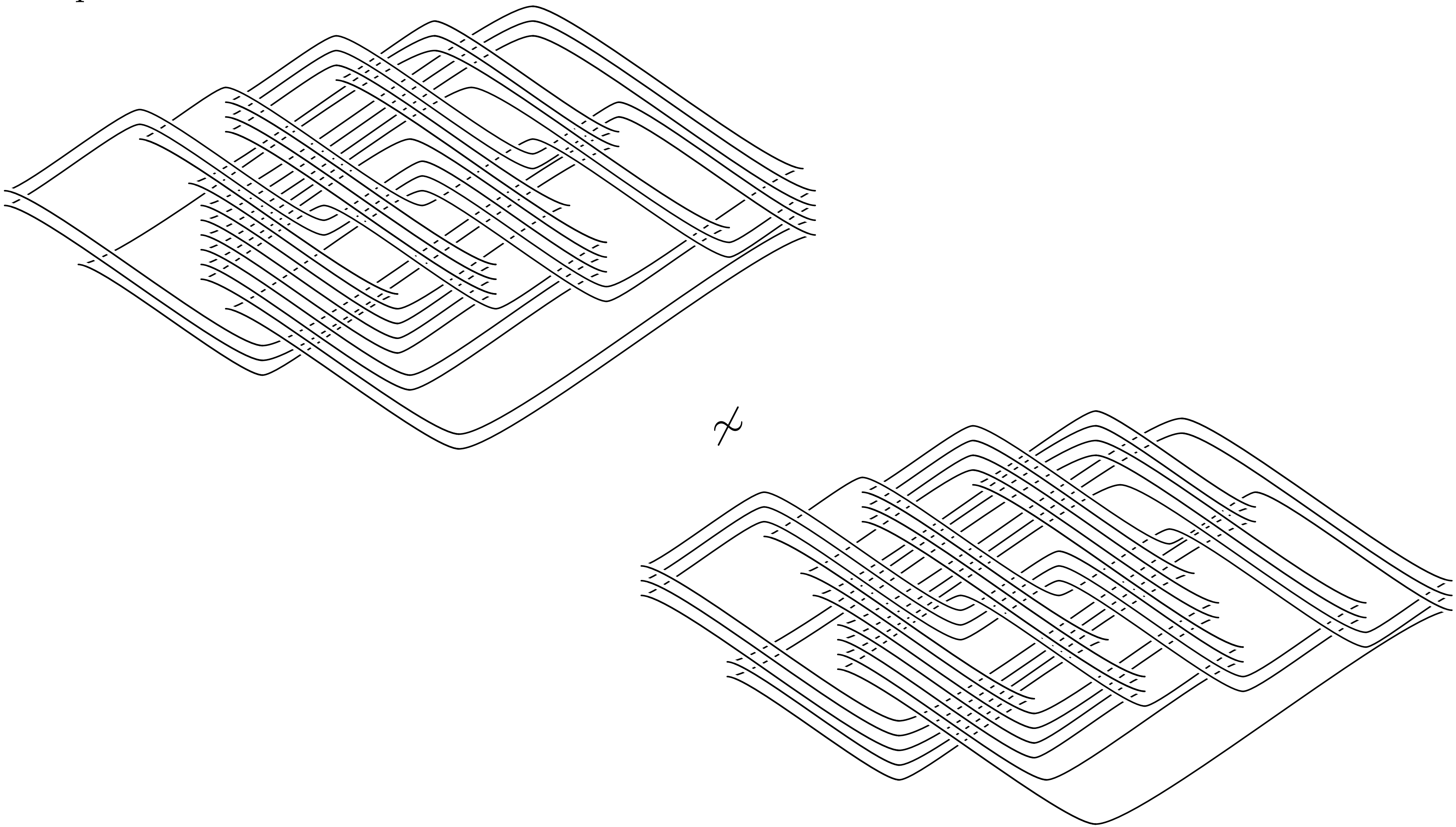
We may assume $[R_1]_{\overrightarrow{\mathrm{II}},\overleftarrow{\mathrm{II}}} = [R_2]_{\overrightarrow{\mathrm{II}},\overleftarrow{\mathrm{II}}}$.

Then if $\mathrm{Sym}(\widehat{R}_1) = \{1\}$ and $[R_1] \neq [R_2]$, then $[R_1]_{\overrightarrow{\mathrm{I}},\overleftarrow{\mathrm{I}}} \neq [R_2]_{\overrightarrow{\mathrm{I}},\overleftarrow{\mathrm{I}}}$

In general, there is an algorithm to find finitely many composable sequences of elementary moves $s_1, \ldots, s_m : R_1 \mapsto R_1$ such that $\widehat{s}_1, \ldots, \widehat{s}_m$ generate $\mathrm{Sym}(\widehat{R}_1)$. From them we learn how many type I stabilizations have to be applied to $[R_1]$ and $[R_2]$ to get the same exchange class provided that $[R_1]_{\overrightarrow{\mathrm{I}},\overleftarrow{\mathrm{I}}} = [R_2]_{\overrightarrow{\mathrm{I}},\overleftarrow{\mathrm{I}}}$.

Corollary: the equivalence of Legendrian links is decidable.

Example

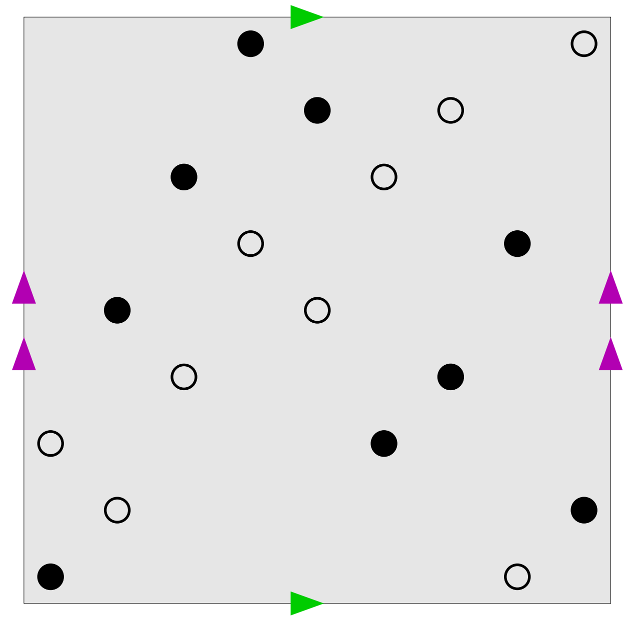


In order to extend the approach to transverse links we have to be able to decide whether or not $[R_1]_{\overrightarrow{\mathrm{II}}} = [R_2]_{\overrightarrow{\mathrm{II}}}$.

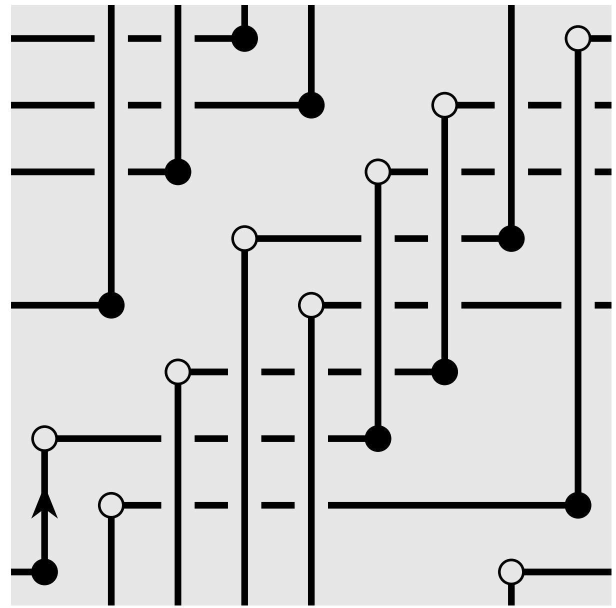
In order to extend the approach to transverse links we have to be able to decide whether or not $[R_1]_{\overrightarrow{\text{II}}} = [R_2]_{\overrightarrow{\text{II}}}$.

This is also decidable.

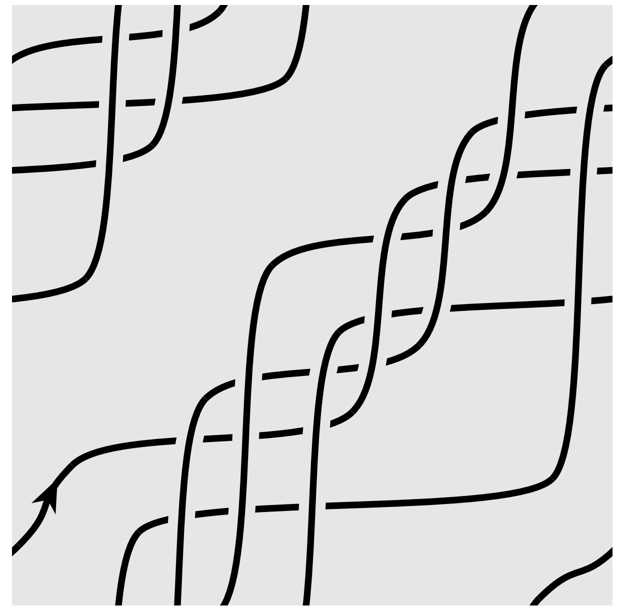
Rectangular diagrams $\longrightarrow$ transverse-Legendrian links



Rectangular diagrams $\longrightarrow$ transverse-Legendrian links

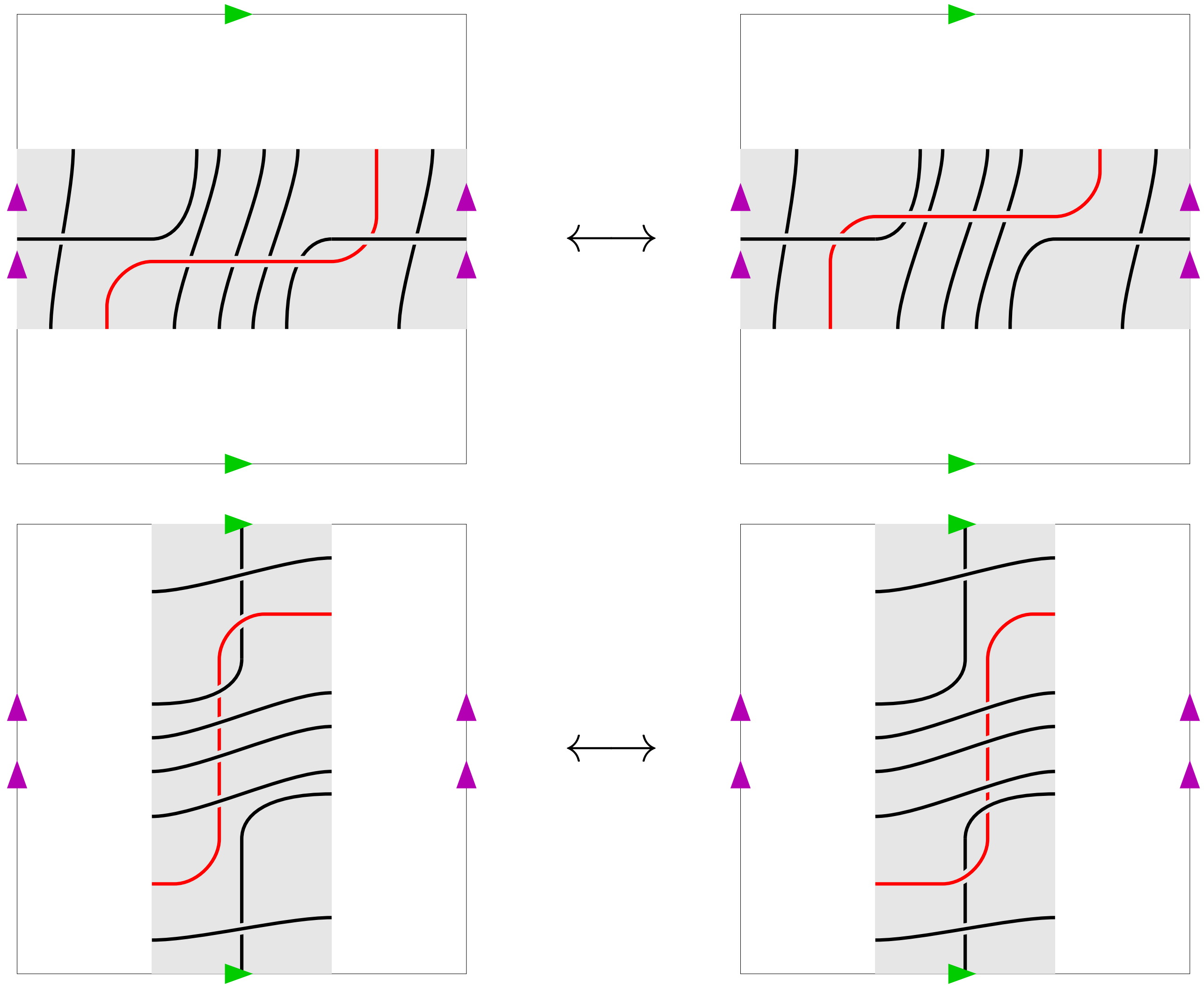


Rectangular diagrams ⟶ transverse-Legendrian links



$$\frac{\{\text{rectangular diagrams}\}}{\text{exchange moves, } S_{\overrightarrow{\mathrm{II}}}} = \frac{\{\text{tranverse-Legendrian link diagrams}\}}{\{\text{Reidemeister-III moves, exchange moves}\}}$$

Bigon moves



The homology class and the number of self-intersections of the diagram are preserved, hence, all combinatorial types of TL-diagrams representing $[R]_{\overrightarrow{\text{II}}}$ are searchable.

The homology class and the number of self-intersections of the diagram are preserved, hence, all combinatorial types of TL-diagrams representing $[R]_{\overrightarrow{\amalg}}$ are searchable.

Corollary: the equivalence of transverse links is decidable.