Trace formulas for the magnetic Laplacian

Yuri A. Kordyukov

Ufa Federal Research Centre, Russian Academy of Sciences

joint work with Iskander A. Taimanov

Magnetic field and twisted symplectic structure

(X, g) a compact Riemannian manifold.
ω an arbitrary closed 2-form on X (a magnetic field),
Define the twisted symplectic form on the phase space $B = T^*X$:

$$\Omega = \Omega_0 + \pi_X^* \omega,$$

where Ω_0 is the canonical symplectic form on T^*X and $\pi_X : T^*X \to X$ is the bundle map.
In local coordinates $(x^1, x^2, \dots, x^n, p_1, p_2, \dots, p_n)$ on T^*X,

$$\Omega_0 = \sum_{j=1}^{n} dp_j \wedge dx^j, \quad \omega = \sum_{j,k=1}^{n} \omega_{jk} dx^j \wedge dx^k,$$

The Hamilton equations of a Hamiltonian $H(x, p)$ with respect to Ω:

$$\frac{dx^j}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial x^j} + \sum_{k=1}^{n} \omega_{jk} \frac{\partial H}{\partial p_k}, \quad j = 1, \dots, n.$$

Magnetic geodesic flow

The magnetic flow $\phi^t : T^*X \to T^*X$ associated with (g, ω) is the Hamiltonian flow of the Hamiltonian

$$h(x,p) = \frac{1}{2}|p|^2_{g^{-1}} = \frac{1}{2}\sum_{j,k=1}^{n} g^{jk} p_j p_k.$$

The Hamilton equations of the magnetic flow ϕ^t:

$$\frac{dx^j}{dt} = \sum_{k=1}^{n} g^{jk} p_k, \quad \frac{dp_j}{dt} = \sum_{k,l=1}^{n} \omega_{jk} g^{kl} p_l, \quad j = 1, \ldots, n.$$

$J : TX \to TX$ a skew-adjoint operator (the Lorenz force):

$$\omega(u,v) = g(Ju, v), \quad u, v \in TX.$$

If $(x(t), \xi(t)) = \phi^t(x, \xi)$ is a trajectory of the magnetic flow ϕ, then its projection to X satisfies the equation

$$\nabla^{TX}_{\dot{x}} \dot{x} = J[\dot{x}],$$

Hermitian line bundle

X a (compact) smooth manifold;
(L, h^L, ∇^L) a Hermitian line bundle on X:

- $L \to X$ a complex line bundle on X:
  locally, over some open $\Omega \subset X$,
  $L|_\Omega \cong \Omega \times \mathbb{C}$; $C^\infty(\Omega, L|_\Omega) \cong C^\infty(\Omega)$;
- h^L a Hermitian structure in the fibers of L:

$$s, s' \in L \to (s, s')_{h^L} \in \mathbb{C},$$

- ∇^L a connection (covariant derivative): for $U \in C^\infty(X, TX)$

$$\nabla^L_U : C^\infty(X, L) \to C^\infty(X, L),$$

  which is Hermitian:

$$\nabla^L_U (s, s')_{h^L} = (\nabla^L_U s, s')_{h^L} + (s, \nabla^L_U s')_{h^L}, \quad s, s' \in C^\infty(X, L).$$

Quantum line bundle

Let (L, h^L, ∇^L) be a Hermitian line bundle on X.
The curvature of ∇^L is the differential two-form R^L on X:

$$R^L(U, V) = \nabla^L_U \nabla^L_V - \nabla^L_V \nabla^L_U - \nabla^L_{[U,V]}, \quad U, V \in TX.$$

Compatibility condition:

$$\omega = iR^L.$$

Quantization condition:

$$(L, h^L, \nabla^L) \text{ exists } \Leftrightarrow [\omega] \in H^2(X, 2\pi\mathbb{Z}).$$

The magnetic Laplacian

Let (L, h^L, ∇^L) be a Hermitian line bundle on X.
The connection can be considered as an operator

$$\nabla^L = d + \Gamma : C^\infty(X, L) \to C^\infty(X, T^*X \otimes L)$$

Fiix a Riemannian metric g on X.
We have L^2-inner products on $C^\infty(X, L)$ and $C^\infty(X, T^*X \otimes L)$:

$$(s, s')_{L^2(X,L)} = \int_X (s(z), s'(z))_{h^L} dv_g(z), \quad s, s' \in C^\infty(X, L).$$

The formally adjoint operator

$$(\nabla^L)^* : C^\infty(X, T^*X \otimes L) \to C^\infty(X, L).$$

For $s \in C^\infty(X, L), s' \in C^\infty(X, T^*X \otimes L)$:

$$(\nabla^L s, s')_{L^2(X,T^*X\otimes L)} = (s, (\nabla^L)^* s')_{L^2(X,L)}.$$

The magnetic Laplacian

Definition

The magnetic Laplacian Δ^L is the Bochner Laplacian associated with a Hermitian line bundle (L, h^L, ∇^L):

$$\Delta^L = (\nabla^L)^*\nabla^L : C^\infty(X, L) \to C^\infty(X, L).$$

Semiclassical parameter:

- $L^p = L^{\otimes p}$ the p-th tensor power of L.
- Δ^{L^p} the magnetic Laplacian acting on $C^\infty(X, L^p)$.
- $\hbar = p^{-1}$ a semiclassical parameter.

Example

- $X = \mathbb{R}^n$, $L = X \times \mathbb{C}$ the trivial Hermitian line bundle.
- The connection form $\Gamma = -i\mathbf{A}$,
  $\mathbf{A} = \sum_{j=1}^d A_j(x) dx_j$ is a real-valued one form
  (the magnetic potential).
- The curvature $R^L = d\Gamma = -id\mathbf{A}$.
- The magnetic field $\omega = \mathbf{B} := d\mathbf{A}$,

$$\mathbf{B} = \sum_{j,k=1}^d B_{jk}(x) dx_j \wedge dx_k, \quad B_{jk} = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}.$$

- The Riemannian metric g is the standard metric on $\mathbb{R}^d$.
- The magnetic Laplacian:

$$\Delta^{L^p} = -\sum_{j=1}^n \left(\frac{\partial}{\partial x_j} - ipA_j(x) \right)^2.$$

The setting

$\{\nu_{p,j}, j \in \mathbb{N}\}$ the eigenvalues of Δ^{L^p}: $\Delta^{L^p} u_{p,j} = \nu_{p,j} u_{p,j}$.

$$\lambda_{p,j} = \sqrt{\nu_{p,j} + p^2}.$$

Fix $E > 1$ and $\varphi \in C_c^\infty(\mathbb{R})$. Define

$$Y_p(\varphi) = \sum_{j=0}^{\infty} \varphi(\lambda_{p,j} - Ep).$$

A smoothed version of $N_p(c) = \#\{j \in \mathbb{N} : |\nu_{p,j} - Ep| < c\}$, $c > 0$.

The main observation:

An asymptotic expansion of $Y_p(\varphi)$ as $p \to \infty$ is expressed in terms of the magnetic geodesic flow (the trace formula).

Magnetic geodesic flow

We will consider the Hamiltonian

$$H(x,p) = (|p|^2_{g^{-1}}+1)^{1/2} = \left(\sum_{j,k=1}^{n} g^{jk}p_jp_k + 1\right)^{1/2}.$$

The corresponding Hamilton equations with respect to the twisted symplectic form $\Omega = \Omega_0 + \pi_X^*\omega$ on T^*X have the form:

$$\frac{dx^j}{dt} = \frac{1}{H}\sum_{k=1}^{n} g^{jk}p_k, \quad \frac{dp_j}{dt} = \frac{1}{H}\sum_{k,l=1}^{n} \omega_{jk}g^{kl}p_l, \quad j=1,\dots,n.$$

The restriction to $B_E = H^{-1}(E)$:

$$\frac{dx^j}{dt} = \frac{1}{E}\sum_{k=1}^{n} g^{jk}p_k, \quad \frac{dp_j}{dt} = \frac{1}{E}\sum_{k,l=1}^{n} \omega_{jk}g^{kl}p_l, \quad j=1,\dots,n.$$

Clean flows

For $E > 1$, $B_E = H^{-1}(E) \subset T^*X$ is a smooth submanifold of T^*X. Assume that the Hamiltonian flow ϕ of H is clean on B_E.

If the set of periods of the flow is discrete, then ϕ is clean on B_E if:

- for every period T, the set $\mathcal{P}_T = \{b \in B_E : \phi^T(b) = b\}$ is a manifold,
- at each $b \in \mathcal{P}_T$ its tangent space $T_b\mathcal{P}_T$ is identical with the set of fixed vectors of $d(\phi^T)_b$.

Guillemin-Uribe trace formula

- $\{\nu_{p,j}, j \in \mathbb{N}\}$ the eigenvalues of Δ^{L^p}.
- $\lambda_{p,j} = \sqrt{\nu_{p,j} + p^2}$.
- For $E > 1$ and $\varphi \in C_c^\infty(\mathbb{R})$, we put

$$Y_p(\varphi) = \sum_{j=0}^{\infty} \varphi(\lambda_{p,j} - Ep).$$

Theorem

The sequence $Y_p(\varphi)$ admits an asymptotic expansion

$$Y_p(\varphi) \sim \sum_{j=0}^{\infty} c_j(p,\varphi) p^{d-j}, \quad p \to \infty,$$

where the coefficients $c_j(p,\varphi)$ are bounded in p.

Guillemin-Uribe trace formula

Theorem (continued)

Moreover, we can say the following about the leading coefficient in the expansion, c_0, and the degree $d = d(\varphi)$:

$$Y_p(\varphi) \sim \sum_{j=0}^{\infty} c_j(p,\varphi)p^{d-j}, \quad p \to \infty,$$

If 0 is the only period in *supp*$(\hat{\varphi})$, then $d = n - 1$ and

$$c_0(p,\varphi) = (2\pi)^{-n}\hat{\varphi}(0)\mathrm{Vol}(B_E),$$

Theorem (continued)

Assume:

- there is a unique period T of the flow in the support of $\hat{\varphi}$.
- Let $Y_1, \dots, Y_r$ be the connected components of $\mathcal{P}_T = \{b \in B_E : \phi^T(b) = b\}$ of maximal dimension k.

Then

$$d = k - 1,$$

and for each j there is a density ν_j on Y_j defined in terms of the classical dynamics:

$$c_0(p, \varphi) = \hat{\varphi}(T) \sum_{r=1}^{r} e^{\pi i \sigma_j / 2} e^{-ipS_j} \int_{Y_j} \nu_j,$$

where σ_j is the (common) Maslov index of the trajectories in Y_j and S_j their (common) action.

The action of periodic trajectory

γ is a periodic trajectory on $B_E \subset T^*X$;
$\pi_X \circ \gamma$ the projection of γ to X;
In the case when $\omega = dA$ for some real-valued 1-form A, the action S_γ of γ:

$$S_\gamma = \pm L\sqrt{E^2 - 1} + \int_\gamma \pi_X^* A.$$

In the case of an arbitrary ω the action of γ is defined modulo multiples of 2π:

$$S_\gamma = \pm L\sqrt{E^2 - 1} + h_A(\gamma).$$

$h_A(\gamma) \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ the holonomy of $\pi_X \circ \gamma$ with respect to ∇^L

Non-degenerate trajectories

Assume:

there is a unique periodic trajectory $\gamma \subset B_E$ whose period T_γ is in the support of $\hat{\varphi}$, $Y_j = \gamma$;

Then

$\mathcal{P}_T = \gamma$, $d = 0$ and the density ν on γ can be computed explicitly:

$$c_0(p,\varphi) = \frac{T_\gamma^{\#} e^{\pi i \sigma_\gamma/2}}{2\pi |I - P_\gamma|^{1/2}} e^{-ipS_\gamma} \hat{\varphi}(T_\gamma)$$

where:

- $T_\gamma^{\#}$ the primitive period of γ;
- P_γ the Poincare map of γ;
- σ_γ the Maslov index of γ.

Constant magnetic field on the two-torus

- $X = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, $g = dx^2 + dy^2$.
- L the line bundle over $\mathbb{T}^2$, whose sections are identified with functions $u \in C^\infty(\mathbb{R}^2)$ such that

$$u(x+1, y) = e^{iBy} u(x,y), \quad u(x, y+1) = u(x,y).$$

- e^{iBy} should be periodic, that gives the quantization condition

$$B = 2\pi n, \quad n \in \mathbb{Z}.$$

- Put

$$B = 2\pi.$$

- The Hermitian connection on L has the form

$$\nabla^L = d - iA,$$

  where A is the magnetic potential: $A = 2\pi x dy$.
- The magnetic form $\omega = dA = 2\pi dx \wedge dy$.

The spectrum

The operator

$$\Delta^{L^p} = -\frac{\partial^2}{\partial x^2} - \left(\frac{\partial}{\partial y} - 2\pi pix\right)^2.$$

The eigenvalue problem

$$\Delta^{L^p} u(x,y) = \lambda u(x,y),$$

$$u(x+1,y) = e^{iBy} u(x,y), u(x,y+1) = u(x,y).$$

We can make a Fourier transform in y, writing

$$u(x,y) = \sum_{k\in\mathbb{Z}} u_k(x) e^{2\pi iky}.$$

Then we have for u_k

$$\left[-\frac{d^2}{dx^2} + 4\pi^2 p^2 \left(x + \frac{k}{p}\right)^2\right] u_k(x) = \lambda u_k(x), \quad u_k(x+1) = u_{k-p}(x)$$

The spectrum

The eigenvalues

$$\nu_{p,j} = 2\pi p(2j+1), \quad j = 0, 1, 2, \dots .$$

The corresponding eigenfunction

$$u_{k,j}(x) = a_k h_k \left(2\pi p \left(x + \frac{k}{p}\right)\right).$$

So we can take arbitrary $a_0, a_1, \dots, a_{p-1}$ and we get the eigenfunction of H:

$$u(x,y) = \sum_{k=0}^{p-1} a_k \sum_{j\in\mathbb{Z}} h_k\left(2\pi(px+k+pj)\right) e^{2\pi i(k-pj)y}.$$

Therefore, the multiplicity of the eigenvalue $\nu_{p,j}$

$$m_{p,j} = p.$$

The trace formula

For $\varphi \in C_c^\infty(\mathbb{R})$, we have

$$Y_p(\varphi) = \sum_{j=0}^{\infty} p\varphi\left(\sqrt{p^2 + 2\pi p(2j+1)} - Ep\right).$$

Let us write

$$j = \frac{E^2-1}{4\pi}p - \left\{\frac{E^2-1}{4\pi}p\right\} + n, \quad n \in \mathbb{Z}.$$

Then we get

$$Y_p(\varphi)$$
$$= \sum_{n=-\left[\frac{E^2-1}{4\pi}p\right]}^{\infty} p\varphi\left(\sqrt{E^2p^2 + 2\pi p(2n+1) - 4\pi p\left\{\frac{E^2-1}{4\pi}p\right\}} - Ep\right).$$

Recall the Taylor expansion for $f(x)=\sqrt{1+x}-1$ at $x=0$:

$$f(x)=\sqrt{1+x}-1=\frac{1}{2}x-\frac{1}{8}x^2+\sum_{k=3}^{\infty}c_kx^k,$$

where

$$c_k=(-1)^{k-1}\frac{(2k-3)!}{2^{2k-2}k!(k-2)!},\quad k\geqslant 2.$$

For each n, we have the following asymptotic expansion as $p\to+\infty$:

$$\begin{aligned}\lambda_{p,j}-Ep=&\frac{1}{E}\left(\pi(2n+1)-2\pi\left\{\frac{E^2-1}{4\pi}p\right\}\right)\\&-\frac{1}{8E^3p}\left(2\pi(2n+1)-4\pi\left\{\frac{E^2-1}{4\pi}p\right\}\right)^2\\&+\sum_{k=3}^{\infty}\frac{c_k}{E^{2k-1}p^{k-1}}\left(2\pi(2n+1)-4\pi\left\{\frac{E^2-1}{4\pi}p\right\}\right)^k.\end{aligned}$$

Using Taylor expansion of φ:

$$\varphi(a_0 + a_1 p^{-1} + a_2 p^{-2} + \dots)$$
$$= \varphi(a_0) + \varphi'(a_0)a_1 + (\varphi'(a_0)a_2 + \frac{1}{2}\varphi''(a_0)a_1^2)p^{-2} + \dots.$$

we obtain an asymptotic expansion for $Y_p(\varphi)$:

$$Y_p(\varphi) \sim c_0(p,\varphi)p + c_1(p,\varphi) + c_0(p,\varphi)p^{-1} + \dots, \quad p \to \infty,$$

For the leading term of order p, we get

$$c_0(p,\varphi) = \sum_{n\in\mathbb{Z}} \varphi\left(\frac{\pi(2n+1)}{E} - \frac{2\pi}{E}\left\{\frac{E^2-1}{4\pi}p\right\}\right).$$

By Poisson summation formula

$$\sum_{n\in\mathbb{Z}} f(nP+t) = \sum_{k\in\mathbb{Z}} \frac{1}{P}\hat{f}\left(\frac{2\pi k}{P}\right) e^{2\pi i \frac{k}{P} t}. \tag{1}$$

with

$$\hat{f}(k) = \int f(x)e^{-ikx}dx,$$

we get $P = \frac{2\pi}{E}$, $t = \frac{\pi}{E} - \frac{2\pi}{E}\left\{\frac{E^2-1}{4\pi}p\right\}$ and, therefore,

$$\begin{aligned} c_0(p,\varphi) &= \sum_{k\in\mathbb{Z}} \frac{E}{2\pi}\hat{\varphi}(kE)e^{ik\pi}e^{-2\pi ik\left\{\frac{E^2-1}{4\pi}p\right\}} \\ &= \frac{E}{2\pi}\sum_{k\in\mathbb{Z}} \hat{\varphi}(kE)e^{ik\pi}e^{-ik\frac{E^2-1}{2}p}. \end{aligned}$$

For the next term, we have

$$c_1(p,\varphi) = -\sum_{n\in\mathbb{Z}} \frac{1}{2E^3}\varphi'\left(\frac{\pi(2n+1)}{E} - \frac{2\pi}{E}\left\{\frac{E^2-1}{4\pi}p\right\}\right) \times$$
$$\times\left(\frac{\pi(2n+1)}{E} - \frac{2\pi}{E}\left\{\frac{E^2-1}{4\pi}p\right\}\right)^2.$$

By Poisson summation formula (with $f=\varphi(x)x^2$)

$$\sum_{n\in\mathbb{Z}}\varphi'(nP+t)(nP+t)^2 = -\sum_{k\in\mathbb{Z}}\left(\frac{2i}{P}\hat{\varphi}'\left(\frac{2\pi k}{P}\right) + \frac{2\pi ik}{P^2}\hat{\varphi}''\left(\frac{2\pi k}{P}\right)\right)e^{2\pi i\frac{k}{P}t}.$$

So we get

$$c_1(p,\varphi) = -\sum_{k\in\mathbb{Z}}\left(\frac{i}{2\pi E^2}\hat{\varphi}'(kE) + \frac{ik}{4\pi E}\hat{\varphi}''(kE)\right)e^{-ik\frac{E^2-1}{2}p}.$$

Geometric interpretation

The magnetic form ω on $\mathbb{T}^2$ given by

$$\omega = B dx \wedge dy, \quad B \in \mathbb{R}.$$

The Hamiltonian H has the form

$$H(x, y, p_x, p_y) = \left(p_x^2 + p_y^2 + 1\right)^{1/2},$$

and the reduced Hamiltonian system on B_E is given by

$$\dot{x} = \frac{1}{E} p_x, \quad \cdot y = \frac{1}{E} p_y, \quad \dot{p}_x = \frac{1}{E} B p_y, \quad \dot{p}_y = -\frac{1}{E} B p_x.$$

This system can be easily solved:

$$
\begin{aligned}
x(t) =& x^0 + \frac{1}{B} p_y^0 \left(1 - \cos \frac{B}{E} t\right) + \frac{1}{B} p_x^0 \sin \frac{B}{E} t, \\
y(t) =& y^0 + \frac{1}{B} p_y^0 \sin \frac{B}{E} t - \frac{1}{B} p_x^0 \left(1 - \cos \frac{B}{E} t\right), \\
p_x(t) =& p_y^0 \sin \frac{B}{E} t + p_x^0 \cos \frac{B}{E} t, \\
p_y(t) =& p_y^0 \cos \frac{B}{E} t - p_x^0 \sin \frac{B}{E} t.
\end{aligned}
$$

So each trajectory is periodic with period $T = \frac{2\pi E}{B}$. Its projection γ is the image of a circle of radius $\frac{1}{B}\sqrt{(p_x^0)^2 + (p_y^0)^2}$ under the natural projection $f : \mathbb{R}^2 \to \mathbb{T}^2$.

Its length is equal to

$$L = 2\pi \frac{1}{B}\sqrt{(p_x^0)^2 + (p_y^0)^2} = 2\pi \frac{1}{B}\sqrt{E^2 - 1}.$$

Next, we compute

$$\int_\gamma A = \int_{f^{-1}(\gamma)} Bxdy.$$

By Stokes formula, we compute (with D, the disc bounded by γ and S, the area of this disc):

$$\int_\gamma A = -\int_D Bdx \wedge dy = -BS = -\frac{\pi}{B}((p_x^0)^2 + (p_y^0)^2) = -\frac{\pi}{B}(E^2 - 1).$$

When $B = 2\pi$, we get

$$S_\gamma = \int_\gamma A + L\sqrt{E^2 - 1} = \frac{1}{2}(E^2 - 1)$$

We have

$$B_E = H^{-1}(E) = \{(x, y, p_x, p_y) : p_x^2 + p_y^2 = E^2 - 1\} \subset T^*X.$$

In the polar coordinates $p_x = R\cos\varphi$, $p_y = R\sin\varphi$, we have

$$dp_x\, dp_y = R\, dR\, d\varphi, \quad dH = \frac{R\, dR}{H},$$

therefore, Liouville measure on B_E is given by

$$\mathrm{Vol} = \frac{dx\, dy\, dp_x\, dp_y}{dH} = E\, d\varphi$$

and

$$\mathrm{Vol}(B_E) = 2\pi E \mathrm{Vol}(X) = 2\pi E.$$

$d = 1$, ϕ_t is clean on B_E (because it is periodic).

$$c_0(p,\varphi)=\frac{E}{2\pi}\sum_{k\in\mathbb{Z}}\hat{\varphi}(kE)e^{ik\pi}e^{-ik\frac{E^2-1}{2}p}.$$

If 0 is the only period in $supp(\hat{\varphi})$, then $d=n-1$ and

$$c_0(p,\varphi)=(2\pi)^{-n}\hat{\varphi}(0)\mathrm{Vol}(B_E).$$

For $k=0$

$$c_0(p,\varphi)=\frac{E}{2\pi}\hat{\varphi}(0)=(2\pi)^{-2}\hat{\varphi}(0)\mathrm{Vol}(B_E).$$

$$\mathrm{Vol}(B_E)=2\pi E.$$

$$c_0(p,\varphi) = \frac{E}{2\pi}\sum_{k\in\mathbb{Z}} \hat{\varphi}(kE) e^{ik\pi} e^{-ik\frac{E^2-1}{2}p}.$$

Assume that there is a unique period T of the flow in the support of $\hat{\varphi}$:

$$c_0(p,\varphi) = \hat{\varphi}(T)\sum_{r=1}^{r} e^{\pi i\sigma_j/2} e^{-ipS_j} \int_{Y_j} \nu_j,$$

where σ_j is the (common) Maslov index of the trajectories in Y_j and S_j their (common) action.
k the multiplicity of the trajectory.
Each trajectory is periodic with period $T = E$.
$S_\gamma = \int_\gamma A + L\sqrt{E^2-1} = \frac{1}{2}(E^2-1)$.
$\sigma = 2$ is the Maslov index.

The two-sphere

Suppose that the manifold X is the round two-sphere

$$x^2 + y^2 + z^2 = R^2.$$

In the spherical coordinates

$$x = R\sin\theta\cos\varphi, y = R\sin\theta\sin\varphi, z = R\cos\theta, \quad \theta \in (0,\pi), \varphi \in (0, 2\pi),$$

the Riemannian metric g on X is given by

$$g = R^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

and the magnetic form ω by

$$\omega = \frac{1}{2} dvol_M = \frac{1}{2}\sin\theta d\theta \wedge d\varphi.$$

For any $\varphi \in C_c^\infty(\mathbb{R})$ and $E > 1$, one has an asymptotic expansion

$$Y_p(\varphi) \sim \sum_{j=0}^{\infty} c_j(p,\varphi)p^{1-j}, \quad p \to \infty.$$

The coefficients c_j can be computed explicitly.
For the first two of them, we get

$$c_0(p,\varphi) = \sum_{k\in\mathbb{Z}} 2ER^2\hat{\varphi}\left(\frac{2\pi ERk}{\sqrt{E^2-1+\frac{1}{4R^2}}}\right) e^{\pi ik(p+1)} e^{-2\pi ikR\sqrt{E^2-1+\frac{1}{4R^2}}p},$$

$$
\begin{aligned}
c_1(p,\varphi) = \sum_{k\in\mathbb{Z}} \Bigg[& 2iR^2 \hat{\varphi}' \left(\frac{2\pi ERk}{\sqrt{E^2 - 1 + \frac{1}{4R^2}}} \right) \\
& - \frac{\pi ER(4R^2-1)k}{2(E^2 - 1 + \frac{1}{4R^2})^{3/2}} i\hat{\varphi}'' \left(\frac{2\pi ERk}{\sqrt{E^2 - 1 + \frac{1}{4R^2}}} \right) \\
& - \frac{\pi iERk}{2\sqrt{E^2 - 1 + \frac{1}{4R^2}}} \hat{\varphi} \left(\frac{2\pi ERk}{\sqrt{E^2 - 1 + \frac{1}{4R^2}}} \right) \Bigg] \times \\
& \times e^{\pi ik(p+1)} e^{-2\pi ikR\sqrt{E^2-1+\frac{1}{4R^2}}p}.
\end{aligned}
$$

In particular, when $R = 1/2$, we get

$$c_0(p,\varphi) = \sum_{k\in\mathbb{Z}} \frac{E}{2}\hat{\varphi}(\pi k)\, e^{\pi ik(p+1)} e^{-\pi ikEp},$$

$$c_1(p,\varphi) = \sum_{k\in\mathbb{Z}} \left[\frac{1}{2} i\hat{\varphi}'(\pi k) - \frac{\pi ik}{4}\hat{\varphi}(\pi k)\right] e^{\pi ik(p+1)} e^{-\pi ikEp}.$$

The hyperbolic plane

Consider the hyperbolic plane $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ equipped with a Riemannian metric

$$g = \frac{1}{y^2}[dx^2 + dy^2].$$

Assume that the line bundle L is trivial and the connection ∇^L on L is given by the connection 1-form

$$A = \frac{B}{y}dx.$$

So we have

$$\omega = B\, dvol_{\mathbb{H}} = B\frac{dx \wedge dy}{y^2}.$$

The hyperbolic plane

Let $\Gamma \subset PSL(2,\mathbb{R})$ be a cocompact lattice acting on $\mathbb{H}$ without fixed points.
The quotient $X = \Gamma \setminus \mathbb{H}$ is a compact Riemannian surface.
The prequantization condition holds if

$$(2g-2)B \in \mathbb{Z},$$

where g is the genus of X.

The hyperbolic plane

The sections of the associated line bundle L on X can be identified with functions ψ on $\mathbb{H}$ such that

$$\psi(\gamma z)=\psi(z)\exp(-i2B\arg(cz+d))=\left(\frac{cz+d}{|cz+d|}\right)^{-2B}\psi(z)$$

for any $z\in\mathbb{H}$ and $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma$.

Let $K=T^*_{(1,0)}X$ be the canonical bundle of holomorphic one-forms on X. Then ψ can be associated with a section $g(z)dz^{-B}\in K^{-B}$, where $g(z):=y^B\psi(z)$.

The function g satisfies

$$g(\gamma z)=(cz+d)^{-2B}g(z),\quad \gamma\in\Gamma.$$

and is called a Γ-automorphic form of weight $-2B$.

The hyperbolic plane

Put $B = 1$. For any $\varphi \in C_c^\infty(\mathbb{R})$ and $1 < E < \sqrt{2}$, one has an asymptotic expansion

$$Y_p(\varphi) \sim \sum_{j=0}^{\infty} c_j(p,\varphi) p^{1-j}, \quad p \to \infty,$$

The coefficients c_j can be computed explicitly.
For the first two of them, we get

$$c_0(p,\varphi) = (2g-2)E \sum_{k\in\mathbb{Z}} \hat{\varphi}\left(\frac{2\pi kE}{\sqrt{2-E^2}}\right) e^{ik\pi} e^{2\pi ik\sqrt{2-E^2}p}.$$

The hyperbolic plane

$$
\begin{aligned}
c_1(p,\varphi) = \Bigg[& \sum_{k\in\mathbb{Z}} (2g-2) 2i \hat{\varphi}' \left(\frac{2\pi kE}{\sqrt{2-E^2}} \right) \\
& + \sum_{k\in\mathbb{Z}} (2g-2) \frac{1}{8} i \frac{2\pi kE}{\sqrt{2-E^2}} \hat{\varphi} \left(\frac{2\pi kE}{\sqrt{2-E^2}} \right) \\
& + \sum_{k\in\mathbb{Z}} (2g-2) i \frac{2\pi kE}{(2-E^2)^{3/2}} \hat{\varphi}'' \left(\frac{2\pi kE}{\sqrt{2-E^2}} \right) \Bigg] e^{ik\pi} e^{2\pi ik\sqrt{2-E^2}p}.
\end{aligned}
$$

The magnetic Katok example

Suppose that the manifold X is the round two-sphere

$$x^2 + y^2 + z^2 = 1.$$

In the spherical coordinates

$$x = \sin\theta\cos\varphi, \quad y = \sin\theta\sin\varphi, \quad z = \cos\theta, \quad \theta \in (0, \pi), \varphi \in (0, 2\pi),$$

the Riemannian metric g on X is given by

$$g = \frac{d\theta^2}{1 - \epsilon^2 \sin^2\theta} + \frac{\sin^2\theta}{(1 - \epsilon^2\sin^2\theta)^2} d\varphi^2,$$

and the magnetic form ω by

$$\omega = dA = \frac{\epsilon \sin 2\theta}{(1 - \epsilon^2\sin^2\theta)^2} d\theta \wedge d\varphi,$$

where

$$A = -\frac{\epsilon \sin^2\theta}{1 - \epsilon^2\sin^2\theta} d\varphi.$$

The trace formula near zero

If 0 is the only period in *supp*($\hat{\varphi}$), then, for $E > 1$, one has an asymptotic expansion

$$Y_p(\varphi) \sim \sum_{j=0}^{\infty} c_j(p,\varphi)p^{1-j}, \quad p \to \infty,$$

where

$$c_0(p,\varphi) = 2E\hat{\varphi}(0). \tag{2}$$

The trace formula away zero

If 0 is not in the support of $\hat{\varphi}$, then, for $E=\sqrt{2}$, one has an asymptotic expansion

$$Y_p(\varphi)\sim\sum_{j=0}^{\infty}c_j(p,\varphi)p^{-j},\quad p\to\infty,$$

where

$$c_0(p,\varphi)=\sum_{k\neq 0}\frac{e^{ik\pi}}{\sqrt{2}(1-\epsilon^2)}\left(\frac{e^{-ipk\frac{2\pi}{1+\epsilon}}}{\sin\frac{\pi k}{1-\epsilon}}+\frac{e^{-ipk\frac{2\pi}{1-\epsilon}}}{\sin\frac{\pi k}{1+\epsilon}}\right)\hat{\varphi}\left(\frac{2\pi\sqrt{2}k}{1-\epsilon^2}\right).$$

The magnetic geodesic flow

The Hamiltonian H has the form

$$H = \left((1-\epsilon^2\sin^2\theta)p_\theta^2 + \frac{(1-\epsilon^2\sin^2\theta)^2}{\sin^2\theta}p_\varphi^2 + 1\right)^{1/2},$$

and the reduced Hamiltonian system on B_E is given by

$$\begin{aligned}
\dot\theta &= \frac{1}{E}(1-\epsilon^2\sin^2\theta)p_\theta,\\
\dot\varphi &= \frac{1}{E}\frac{(1-\epsilon^2\sin^2\theta)^2}{\sin^2\theta}p_\varphi,\\
\dot p_\theta &= \frac{\epsilon^2}{E}\sin\theta\cos\theta p_\theta^2 + \frac{1}{E}\left[\frac{\cos\theta}{\sin^3\theta} - \epsilon^4\sin\theta\cos\theta\right]p_\varphi^2 + \frac{2\epsilon}{E}\cot\theta p_\varphi,\\
\dot p_\varphi &= -\frac{\epsilon}{E}\frac{\sin 2\theta}{1-\epsilon^2\sin^2\theta}p_\theta.
\end{aligned}$$

The magnetic geodesic flow

This system is integrable with an additional first integral given by

$$P = p_\varphi + \epsilon \frac{\sin^2 \theta}{1 - \epsilon^2 \sin^2 \theta}.$$

It is easy to check that this system has two periodic solutions

$$\theta(t) = \frac{\pi}{2}, \quad \varphi(t) = \pm \frac{\sqrt{E^2 - 1}}{E}(1 - \epsilon^2)t + \varphi_0,$$

$$p_\theta(t) = 0, \quad p_\varphi(t) = \pm \frac{\sqrt{E^2 - 1}}{1 - \epsilon^2}.$$

The period and the length of the corresponding periodic trajectory $\ell_\pm$

$$T = \frac{2\pi E}{(1 - \epsilon^2)\sqrt{E^2 - 1}}, \quad L = \frac{2\pi}{1 - \epsilon^2}.$$

The magnetic geodesic flow

It turns out that, if $E = \sqrt{2}$ and ϵ is irrational, these are the only periodic orbits of the magnetic geodesic flow. Indeed, first of all, observe that the restriction of the Hamiltonian system to the energy level B_E is described by the Lagrangian

$$L(\theta, \varphi, \dot{\theta}, \dot{\varphi}) = \sqrt{E^2 - 1}\frac{\sqrt{(1 - \epsilon^2 \sin^2 \theta)\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2}}{(1 - \epsilon^2 \sin^2 \theta)^2} - \frac{\epsilon \sin^2 \theta}{1 - \epsilon^2 \sin^2 \theta}\dot{\varphi}.$$

The desired statement is proved in Rademacher04, where it is shown that this is exactly the Finsler metric introduced be A. Katok. These metrics coincide with the examples of constant flag curvature Finsler metric on S^2 given by Z. Shen.