Sphere packings, kissing numbers and related problems

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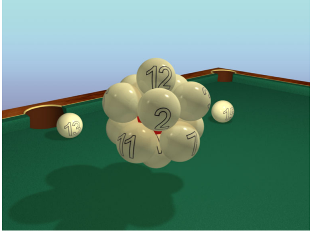
Abstract

The kissing number problem asks for the maximal number $k(n)$ of equal size nonoverlapping spheres in n-dimensional space that can touch another sphere of the same size. This problem in dimension three was the subject of a famous discussion between Isaac Newton and David Gregory in 1694. In three dimensions the problem was finally solved only in 1953 by Schütte and van der Waerden. In this talk we are going to give an overview of this problem, and to present our solution of a long-standing problem about the kissing number in four dimensions. Namely, the equality $k(4) = 24$ is proved.



References

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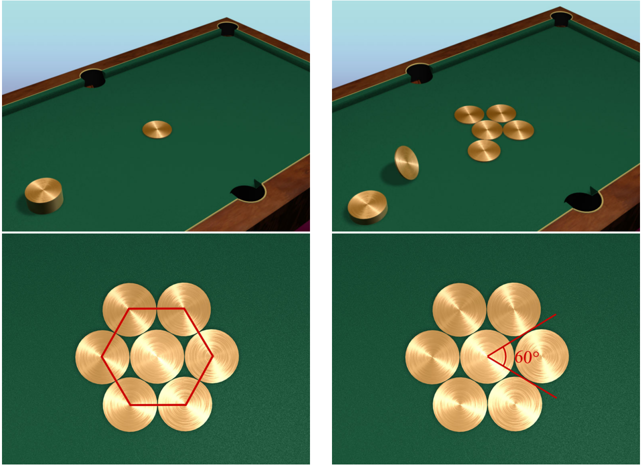
F. Pfender and G.M. Ziegler, Kissing numbers, sphere packings, and some unexpected proofs // Notices Amer. Math. Soc., 51 (2004), 873-883. [2006 Chauvenet Prize winner article]

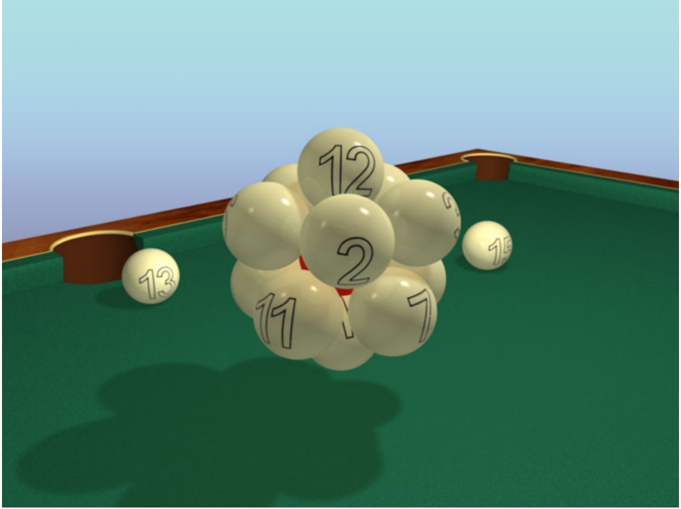
In two dimensions the kissing number problem asks how many coins can touch one given coin at the same time if the coins have the same size. If you arrange the coins on a pool table, it is easy to see that the answer is exactly six.

$$k(2) = 6$$

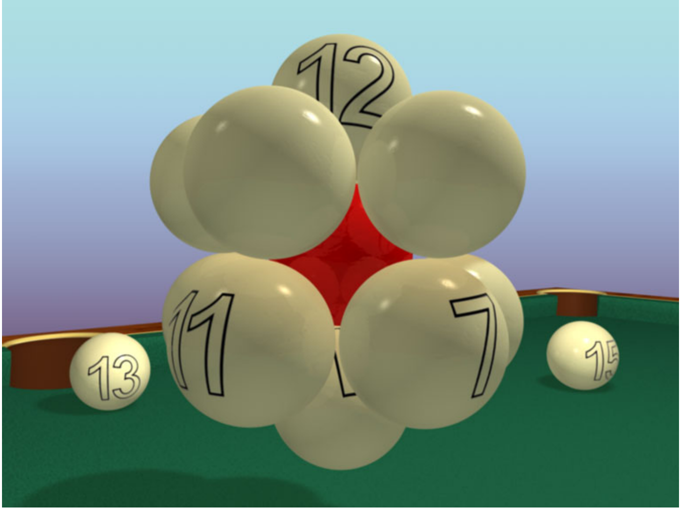


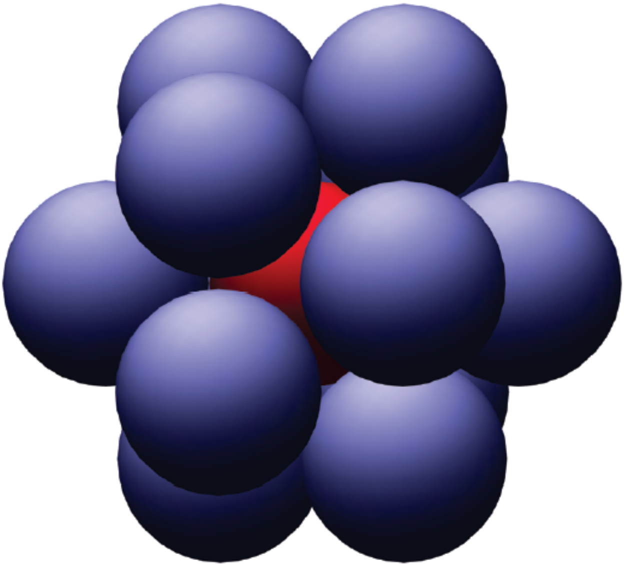
In three dimensions the kissing number problem is asking how many white billiard balls can kiss (touch) a red ball.

The most symmetrical configuration, 12 billiard balls around another, is achieved if the 12 balls are placed at positions corresponding to the vertices of a regular icosahedron.



In the famous book “Regular Polytopes” Coxeter shows that there is so much “play” that it is possible to achieve any permutation of the 12 balls by cleverly rolling them around the central ball in such a way that they never overlap.





(Graphics: Detlev Stalling, ZIB Berlin)

History: Thomas Harriot (1585)

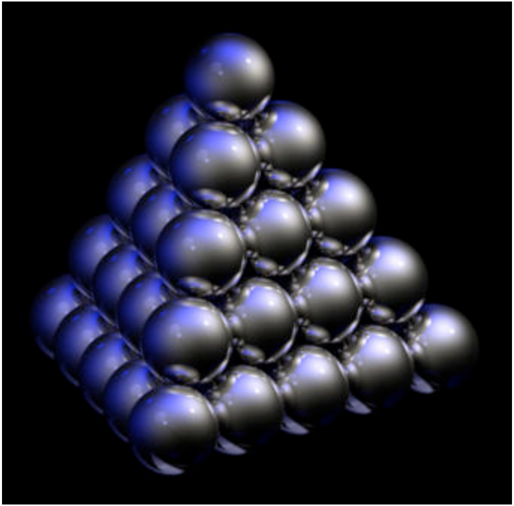
Sir Walter Raleigh's problem:

To develop a formula that would allow to know how many cannonballs can be in a given stack simply by looking at the shape of the pile.

Harriot discovered that for sufficiently large pile the highest density gives the so-called *face centered cubic* (FCC) packing. For the FCC packing the density is:

$$\frac{\pi}{3\sqrt{2}} \approx 0.74048$$

Face Centered Cubic (FCC) packing



History: Johannes Kepler (1611)

J. Kepler. The Six-Cornered Snowflake, 1611

In this little booklet Kepler examined several questions:

– Why honeycomb are formed as hexagon?

– Why the seeds of pomegranates are shaped as dodecahedra?

– Why the petals of flowers are most often grouped in fives?

– Why snowflakes are shaped as they are?

The Kepler Conjecture (1611):

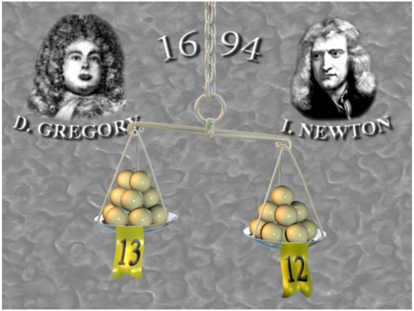
The highest density of a packing of 3*-space by equal spheres* $= 0.74048\ldots$

Hilbert's Problem 18:3 (1900):

"How can one arrange an infinite number of equal solids, of given form, most densely in space, e.g., spheres with given radii. . . How can one fit them together in a manner such that the ratio of the filled space to the unfilled space be as great as possible?"

History: Gregory vs. Newton (1694)

On May 4, 1694 David Gregory paid a visit to Cambridge for several days nonstop discussions about scientific matters with the leading scientist of the day Isaac Newton. Gregory making notices of everything that great master uttered. One of the points discussed, *number* 13, in Gregory's memorandum was 13 spheres problem.
Newton: $k(3) = 12$ vs.
Gregory: $k(3) = 13$ (The main Gregory argument was: area of the unit sphere $\approx$ 14.9$\times$ area of a spherical cap of radius 30°.)



The Newton – Gregory problem = The thirteen spheres problem

History: Gregory vs. Newton -2

Rob Kusner, Wöden Kusner, Jeffrey C. Lagarias, Senya Shlosman: *Configuration Spaces of Equal Spheres Touching a Given Sphere: The Twelve Spheres Problem,* "New Trends in Intuitive Geometry", 219–277, Springer, 2018

In 1694 Isaac Newton and David Gregory had a discussion of touching spheres related to preparing a second edition of Newton's "Principia" ...

Gregory [1694]: *To discover how many stars there are of a given magnitude, he [Newton] considers how many spheres, nearest, second from them, third etc. surround a sphere in a space of three dimensions, there will be 13 of first magnitude,* 4×13 *of second,* $9 \times 4 \times 13$ *of third ...*

Newton [1713], Gregory [1704], ..., Gregory [1715] ...

Thus Gregory expressed a definite opinion that 13 spheres might touch.

History: Lattice sphere packing

Carl Friedrich Gauss (1831): *The FCC packing is the unique densest lattice sphere packing for dimension three.* In particular, $k^*(3) = 12$, where $k^*(n)$ is the max kissing number for lattice packings.

Hérmit (1850,1874); **Lebesgue** (1856); **Selling** (1874); **Minkowski** (1883), . . . , **Mahler** (1992).

Korkine & Zolotareff: $n = 4$ (1872), $n = 5$ (1877).

Blichfeldt (1925, 1929, 1935): $n = 6, 7, 8$.

Cohn & Kumar (2003): $n = 24$.

History: Packing in the plane

Axel Thue provided the first proof that this was optimal in 1890, showing that

The hexagonal lattice is the densest of all possible circle packings, both regular and irregular.

However, his proof was considered by some to be incomplete. The first rigorous proof is attributed to László Fejes Tóth in 1940.

History: Hales (1998)

In 1998, **Thomas Callister Hales,** following the approach suggested by **László Fejes Tóth** in 1953, announced a proof of the Kepler conjecture. Hales' proof is a proof by exhaustion involving checking of many individual cases using complex computer calculations. On 10 August 2014 Hales announced the completion of a formal proof using automated proof checking, removing any doubt.

History: Viazovska (2016)

In 2016, **Maryna Viazovska** announced a proof that the E_8 lattice provides the optimal packing in eight-dimensional space, and soon afterwards she and a group of collaborators (**Cohn, Kumar, Miller, Radchenko**) announced a similar proof that the Leech lattice is optimal in 24 dimensions.

History: Schütte & v. d. Waerden (1953)

Reinhold Hoppe thought he had solved the thirteen spheres problem in 1874. However, there was a mistake — an analysis of this mistake was published by **Thomas Hales**: *The status of the Kepler conjecture*, Mathematical Intelligencer, 16 (1994), 47-58.
Finally, the thirteen spheres problem was solved by **Kurt Schütte** and **Baartel Leendert van der Waerden** in 1953. They had proved:

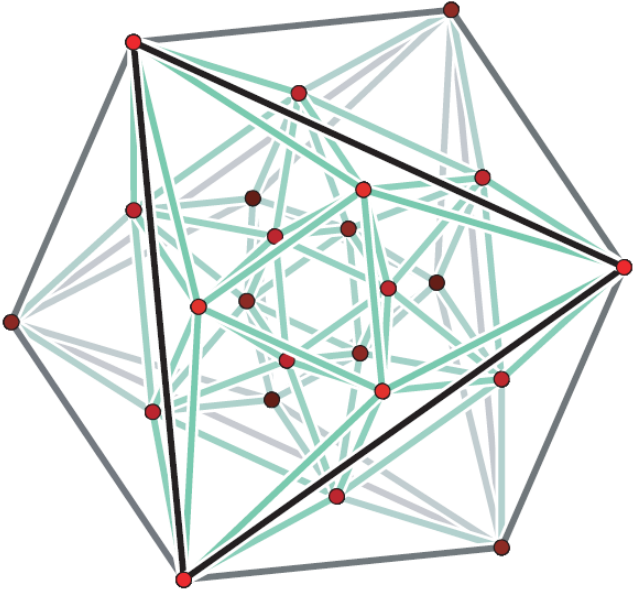
$$k(3) = 12.$$

It's not the end of the story about 13 spheres ...

John Leech(1956) : two-page sketch of a proof $k(3) = 12$.
... It also misses one of the old chapters, about the "problem of the thirteen spheres," whose turned out to need details that we couldn't complete in a way that would make it brief and elegant.
Proofs from THE BOOK, M. Aigner, G. Ziegler, 2^{nd} edition.
W. –Y. Hsiang (2001);
H. Maehara (2001, 2007);
K. Böröczky (2003);
K. Anstreicher (2004);
M. (2006)

k(4) is not less than 24

$n = 4$: There are 24 vectors with two zero components and two components equal to ± 1; they all have length $\sqrt{2}$ and a minimum distance of $\sqrt{2}$. Properl rescaled (that is, multiplied by $\sqrt{2}$), they yield the centers for a kissing configuration of unit spheres and imply that $k(4) \geq 24$. The convex hull of the 24 points yields a famous 4-dimensional polytope, the "24-cell", discovered in 1842 by Ludwig Schäfli. Its facets are 24 regular octahedra.



(Graphics: Michael Joswig/polymake [13])

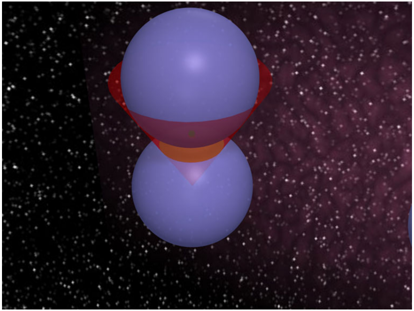
History: Coxeter (1963)

Coxeter proposed upper bounds on $k(n)$ in 1963 for $n = 4, 5, 6, 7$, and 8 these bounds were 26, 48, 85, 146, and 244, respectively.

Coxeter's bounds are based on the conjecture that equal size spherical caps on a sphere can be packed no denser than packing where the Delaunay triangulation with vertices at the centers of caps consists of regular simplices. This conjecture has been proved by **Böröczky** in 1978.

Packing by spherical caps

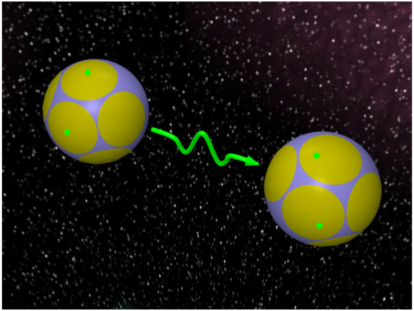
If unit spheres kiss the unit sphere S, then the set of kissing points is the arrangement on S such that the angular distance between any two points is at least 60°. Thus, the kissing number is the maximal number of nonoverlapping spherical caps of radius 30° on S.



Spherical codes

C. E. Shannon in ["A mathematical theory of communication",1948] proposed to apply packings of the unit spheres by spherical caps of given radius r for *coding theory*.

The main application of this theory is in the design of signals for data transmission and storage.



Delsarte's method

Ph. **Delsarte** (1972); **V. M. Sidelnikov** (1974)
Delsarte, **Goethals** and **Seidel** (1975, 1977)

Theorem (Delsarte et al)

If

$$f(t) = \sum_{k=0}^{d} c_k G_k^{(n)}(t)$$

is nonnegative combination of Gegenbauer polynomials, with $c_k \geq 0$ *and* $c_0 > 0$*, and if* $f(t) \leq 0$ *holds for all* $t \in [-1, \frac{1}{2}]$*, then the kissing number in* n *dimensions is bounded by*

$$k(n) \leq \frac{f(1)}{c_0}$$

K(8)=240; k(24)=196560

G.A. Kabatiansky and **V.I. Levenshtein** (1978):

$$2^{0.2075n(1+o(1))} \leq k(n) \leq 2^{0.401n(1+o(1))}$$

In 1979: **V. I. Levenshtein** and independently **A. Odlyzko** and **N.J.A. Sloane** using Delsarte's method have proved that $k(8) = 240$, and $k(24) = 196560$.

Odlyzko & Sloane: upper bounds on $k(n)$ for $n = 4$, 5, 6, 7, and 8 are 25, 46, 82, 140, and 240, respectively.

1979 – 2003

1993: **W.-Y. Hsiang** claims a proof of $k(4) = 24$ (as well as a proof of Kepler's conjecture). His work has not received yet a positive peer review.

1999: **V.V. Arestov** and **A.G. Babenko** proved that the bound $k(4) < 26$ cannot be improved using Delsarte's method.

The kissing problem in four dimensions

$$f_4(t) = 53.76t^9 - 107.52t^7 + 70.56t^5 + 16.38t^4 - 9.83t^3 - 4.12t^2 + 0.434t - 0.016$$

Lemma

Let $P = \{p_1, \ldots, p_m\}$ be unit vectors in $\mathbb{R}^4$ (i.e. points on the unit sphere S^3). Then

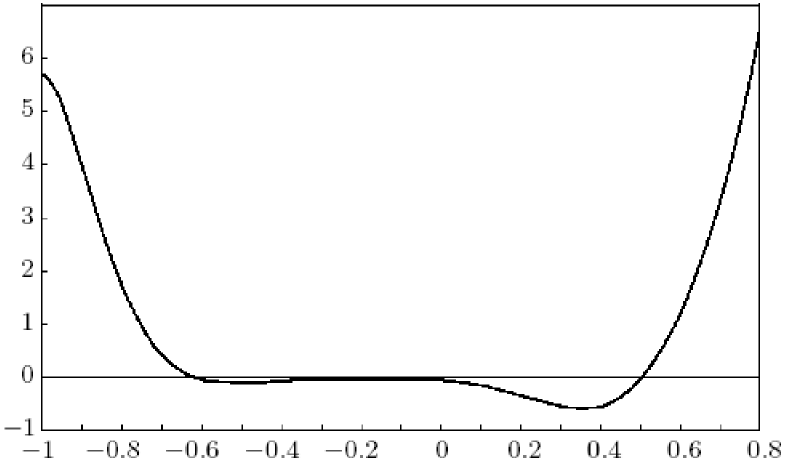
$$S(P) = \sum_{k,\ell} f_4(p_k \cdot p_\ell) \geq m^2.$$

Lemma

Let $P = \{p_1, \ldots, p_m\}$ be a kissing arrangement on the unit sphere S^3 (i.e. $p_k \cdot p_\ell \leq \frac{1}{2}$). Then

$$S(P) = \sum_{k,\ell} f_4(p_k \cdot p_\ell) < 25m.$$

The graph of $y = f_4(t)$



Theorem

$$k(4) = 24$$

Proof.

Suppose P is a kissing arrangement on S^3 with $m = k(4)$. Then P satisfies the assumptions in the lemmas. Therefore,

$$25m > S(P) \geq m^2$$

From this $m < 25$ follows, i.e. $m \leq 24$. From the other side: $m \geq 24$, showing that

$$m = k(4) = 24$$

□

Status 2018: Kissing numbers

The only exact values of kissing numbers known:

n	lattice	regular polytope
$k(1) = 2$	A_1	
$k(2) = 6$	A_2	hexagon
$k(3) = 12$	H_3	icosahedron
$k(4) = 24$	$?D_4$	? 24-cell
$k(8) = 240$	E_8	
$k(24) = 196560$	Λ_{24}	

The four dimensional lattice packing D_4

The checkerboard lattice $D_n := \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : x_1 + \ldots + x_n \text{ even}\}$

The Voronoi cell of D_4 is the regular 24-cell

The density $\Delta_4 = \pi^2/16 = 0.6169...$

The densest packing by unit spheres in four dimensions is conjectured to be the D_4

The center density $\mathrm{CD}_n = \Delta_n / V_n$,
$V_n =$ the volume of a unit n–dimensional sphere.

$\mathrm{CD}_4 = 0.12500$;
Cohn–Elkies bound $= 0.13126$;
de Laat – de Oliveira Filho – Vallentin $= 0.130587$

dim=4: uniqueness of the maximal kissing arrangement

LP bound [Odlyzko & Sloane; Arestov & Babenko] = 25.558...

M. (2003): $k(4) < 24.865$

C. Bachoc & F. Vallentin (2008): $S_7(4) = 24.5797...$

H. D. Mittelmann & F. Vallentin (2010)
$S_{11}(4) = 24.10550859...$
$S_{12}(4) = 24.09098111...$
$S_{13}(4) = 24.07519774...$
$S_{14}(4) = 24.06628391...$

F.C. Machado & F.M. de Oliveira Filho (2018+)
$S_{15}(4) = 24.062758...$
$S_{16}(4) = 24.056903...$

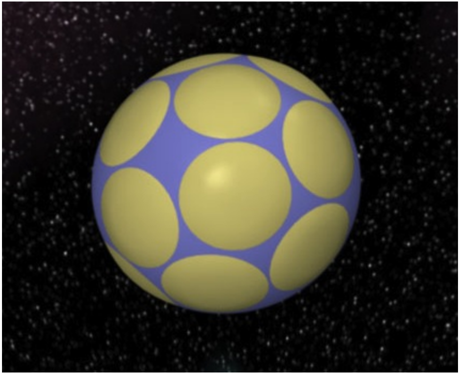
The 24–cell conjecture

Consider the Voronoi decomposition of any given packing P of unit spheres in $\mathbb{R}^4$. The minimal volume of any cell in the resulting Voronoi decomposition of P is at least as large as the volume of a regular 24–cell circumscribed to a unit sphere.

O. R. Musin, Towards a proof of the 24–cell conjecture // Acta Math Hungar., 155 (2018), 184–199

Tammes' problem

Tammes' problem. How must N congruent non-overlapping spherical caps be packed on the surface of a unit sphere so that the angular diameter of spherical caps will be as great as possible



Tammes' problem

It is named after a Dutch botanist who posed the problem in 1930 while studying the distribution of pores on pollen grains. [Tammes P.M.L., *"On the origin of number and arrangement of the places of exit on pollen grains".* Diss. Groningen, 1930.]

This question is also known as the problem of the "inimical dictators":

Where should N dictators build their palaces on a planet so as to be as far away from each other as possible?

Papers

O. R. Musin and A. S. Tarasov, *The strong thirteen spheres problem,* DCG, **48** (2012) 128–141.

O. R. Musin and A. S. Tarasov, *Enumeration of irreducible contact graphs on the sphere,* J. of Math Sciences, **203** (2014), 837–850

O. R. Musin and A. S. Tarasov, *Extreme problems of circle packings ...,* Proc. Steklv. Inst.,**288** (2015), 117–131

O. R. Musin and A. S. Tarasov, *The Tammes problem for N=14,* Experimental Math., **24:4** (2015), 460–468

O. R. Musin and A. V. Nikitenko, *Optimal packings of congruent circles on a square flat torus,* DCG, **55:1** (2016), 1–20.

Packing spheres by spheres: Methods

I. *Area inequalities.* L. Fejes Tóth (1943); for $d > 3$ Coxeter (1963) and Böröczky (1978)

II. *Distance and irreducible graphs.* Schütte, and van der Waerden (1951); Danzer (1963); Leech (1956);...

III. *LP and SDP.* Delsarte et al (1977); Kabatiansky and Levenshtein (1978);...

The Tammes problem

L. Fejes Tóth (1943): $N = 3, 4, 6, 12, \infty$

K. Schütte, and B. L. van der Waerden (1951): $N = 5, 7, 8, 9$

L. Danzer (1963): $N = 10, 11$

R. M. Robinson (1961): $N = 24$

M. & Tarasov: $N = 13$ and $N = 14$

Optimal packings of circles on a square flat torus

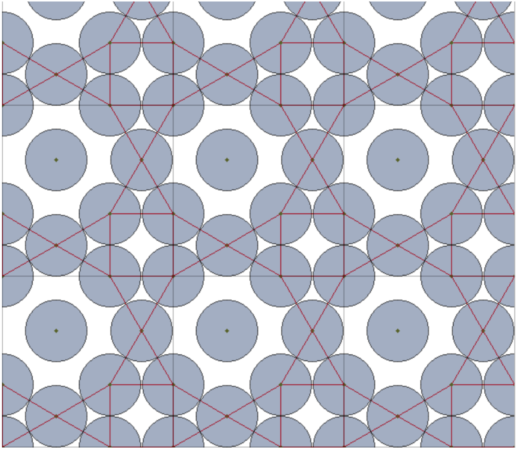
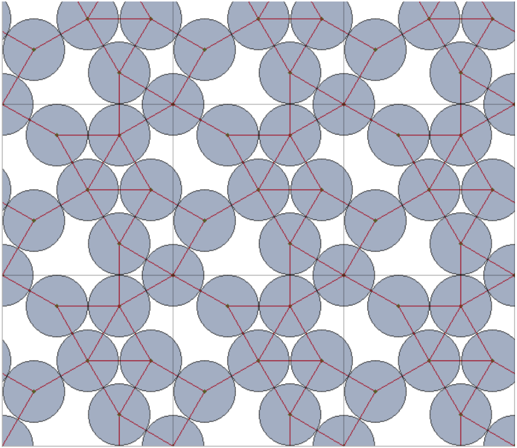


Figure: The first optimal configurations for N=7

Optimal packings of circles on a square flat torus



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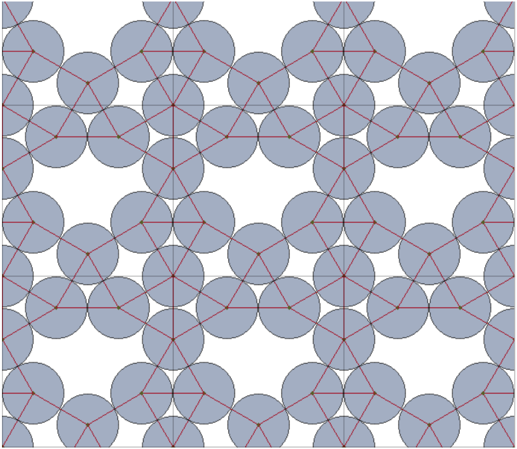

Figure: The third optimal configurations for N=7

Thank you