On the properties of completeness type of function spaces

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Introduction

• A space is *meager* (or *of the first Baire category*) if it can be written as a countable union of closed sets with empty interior.

• A topological space X is *Baire* if the Baire Category Theorem holds for X, i.e., the intersection of any sequence of open dense subsets of X is dense in X.

Introduction

• A space is *meager* (or *of the first Baire category*) if it can be written as a countable union of closed sets with empty interior.

• A topological space X is *Baire* if the Baire Category Theorem holds for X, i.e., the intersection of any sequence of open dense subsets of X is dense in X.

Clearly, if X is a Baire space, then X is not meager. The reverse implication is in general not true. However, it holds for every homogeneous space X (D. Lutzer, R. McCoy, Category in function spaces).

Introduction

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Then a paper (D. Lutzer, R. McCoy) appeared, where various aspects of this topic were considered. In (D. Lutzer, R. McCoy, Category in function spaces.), necessary and, in some cases, sufficient conditions on a space X were obtained under which the space $C_p(X)$ of all continuous real-valued functions $C(X)$ on a space X with the topology of pointwise convergence is Baire.

Introduction

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Theorem (Pytkeev-Tkachuk-van Douwen)

The space $C_p(X)$ is Baire if and only if every pairwise disjoint sequence of non-empty finite subsets of X has a strongly discrete subsequence.

A collection $\mathcal{G}$ of subsets of X is *discrete* if each point of X has a neighborhood meeting at most one element of $\mathcal{G}$, and is *strongly discrete* if for each $G \in \mathcal{G}$ there is an open superset U_G of G such that $\{U_G : G \in \mathcal{G}\}$ is discrete.

Problem Banakh-Gabriyelyan

One of the interesting problems for the space of Baire functions is the Banakh-Gabriyelyan problem (Problem 1.1 in T. Banakh, S. Gabriyelyan, Baire category of some Baire type function spaces, Topology and its Applications, **272** (2020), 107078):

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One of the interesting problems for the space of Baire functions is the Banakh-Gabriyelyan problem (Problem 1.1 in T. Banakh, S. Gabriyelyan, Baire category of some Baire type function spaces, Topology and its Applications, **272** (2020), 107078):

Let α be a countable ordinal. Characterize topological spaces X and Y for which the function space $B_\alpha(X, Y)$ is Baire.

Theorem. (T. Banakh, S. Gabriyelyan, 2020)

$B_\alpha(X,\mathbb{R})$ is Baire for any space X and every countable ordinal $\alpha \geq 2$.

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$B_1(X,\mathbb{R})$?

A G_δ-subset of X containing x is called a G_δ *neighborhood* of x.

Definition

A set $A \subseteq X$ is called *strongly G_δ-disjoint*, if there is a pairwise disjoint collection $\{F_a : F_a \text{ is a } G_\delta \text{ neighborhood of } a,\ a \in A\}$ such that $\{F_a : a \in A\}$ is a *completely G_δ-additive system*, i.e. $\bigcup\limits_{b \in B} F_b \in G_\delta$ for each $B \subseteq A$.

A disjoint sequence $\{\Delta_n : n \in \mathbb{N}\}$ of (finite) sets is *strongly G_δ-disjoint* if the set $\bigcup\{\Delta_n : n \in \mathbb{N}\}$ is strongly G_δ-disjoint.

Theorem 1.

For a space X the following assertions are equivalent:
1. $B_1(X)$ is meager;
2. there is a pairwise disjoint sequence of non-empty finite subsets of X no subsequence of which is strongly G_δ-disjoint.

Main results

Since a non-meager space $B_1(X)$ is Baire, we have the following result.

Theorem 1'.

Let X be a topological space. The following assertions are equivalent:
1. $B_1(X)$ is Baire;
2. every pairwise disjoint sequence of non-empty finite subsets of X has a strongly G_δ-disjoint subsequence.

Let $B_1(X)$ be a Baire space and $Y \subseteq X$. Then $B_1(Y)$ is Baire.

It is well-known that there are Baire spaces X and Y such that $X \times Y$ is not Baire [Ox]. For the product $\prod_{\alpha \in A} B_1(X_\alpha)$ we have the following result.

Theorem 2

If $B_1(X_\alpha)$ is Baire for all $\alpha \in A$, then $\prod_{\alpha \in A} B_1(X_\alpha)$ is Baire.

Main results

Theorem 3.

Let X be a space of countable pseudocharacter. A space $B_1(X)$ is Baire if, and only if, every pairwise disjoint sequence $\{\Delta_i : i \in \mathbb{N}\}$ of non-empty finite subsets of X has a subsequence $\{\Delta_{i_k} : k \in \mathbb{N}\}$ such that $\bigcup\{\Delta_{i_k} : k \in \mathbb{N}\}$ is G_δ.

Main results

Theorem 3.

Let X be a space of countable pseudocharacter. A space $B_1(X)$ is Baire if, and only if, every pairwise disjoint sequence $\{\Delta_i : i \in \mathbb{N}\}$ of non-empty finite subsets of X has a subsequence $\{\Delta_{i_k} : k \in \mathbb{N}\}$ such that $\bigcup\{\Delta_{i_k} : k \in \mathbb{N}\}$ is G_δ.

Theorem 4.

If X is metrizable and $B_1(X)$ is a Baire space, then each separable subset of X without isolated points is meager (of first category) in itself.

Main results

Theorem 3.

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Theorem 4.

If X is metrizable and $B_1(X)$ is a Baire space, then each separable subset of X without isolated points is meager (of first category) in itself.

Corollary 1.

If X is metrizable and separable such that $B_1(X)$ is Baire, then X is meager.

Let us recall that a cover $\mathcal{U}$ of a set X is called

- an ω-cover if each finite set $F \subseteq X$ is contained in some $U \in \mathcal{U}$;
- a γ-cover if for any $x \in X$ the set $\{U \in \mathcal{U} : x \notin U\}$ is finite.

Definition (Gerlits and Nagy)

A topological space X is called a γ-*space* if each countable open ω-cover $\mathcal{U}$ of X contains a γ-subcover of X.

Theorem 5.

Let X be a γ-space. Then $B_1(X)$ is Baire.

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Corollary 2

Let $C_p(X)$ be a Fréchet-Urysohn space. Then $B_1(X)$ is Baire.

On the other hand, when X is a Lindelöf scattered space or a Lindelöf P-space, $C_p(X)$ has the Fréchet-Urysohn property. Thus, we get the following corollaries.

Corollary 3

Let X be a scattered Lindelöf space. Then $B_1(X)$ is Baire.

Corollary 4

Let X be a Lindelöf P-space. Then $B_1(X)$ is Baire.

Choquet property

Both Baire and meager space have game characterizations due to Oxtoby.

The game $G_I(X)$ is started by the player ONE who selects a nonempty open set $V_0 \subseteq X$. Then the player TWO responds selecting a nonempty open set $V_1 \subseteq V_0$. At the n-th inning the player ONE selects a nonempty open set $V_{2n} \subseteq V_{2n-1}$ and the player TWO responds selecting a nonempty open set $V_{2n+1} \subseteq V_{2n}$. At the end of the game, the player ONE is declared the winner if $\bigcap_{n \in \omega} V_n$ is empty. In the opposite case the player TWO wins the game $G_I(X)$.

White, 1975, *weakly α-favorable spaces*

A topological space X is defined to be *Choquet* if the player TWO has a winning strategy in the game $G_I(X)$.

The sequence $\{\mathcal{C}_n : n \in \mathbb{N}\}$ is called *pseudocomplete* if, for any family $\{U_n : n \in \mathbb{N}\}$ such that $\overline{U_{n+1}} \subseteq U_n$ and we have $U_n \in \mathcal{C}_n$ for each $n \in \mathbb{N}$, we have $\bigcap\{U_n : n \in \mathbb{N}\} \neq \emptyset$.

A space X is called *pseudocomplete* if there is a pseudocomplete sequence $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of π-bases in X.

It is a well-known that any pseudocomplete space is Baire and any Čech-complete space is pseudocomplete. Note that if X has a dense pseudocomplete subspace (in particular, if X has a dense Čech-complete subspace) then X is pseudocomplete.

Pseudocompleteness and Choquet for space of Baire-one functions

Theorem 7.

For a space X the following assertions are equivalent:

1. $B_1(X)$ is pseudocomplete;
2. $B_1(X)$ is ω-full in $\mathbb{R}^X$;
3. $B_1(X)$ is G_δ-dense in $\mathbb{R}^X$;
4. $B_1(X)$ is Choquet;
5. Every countable subset of X is strongly G_δ-disjoint.

Recall that a topological space X is called a λ-*space* if every countable subset is of type G_δ in X. A subset X of the real line $\mathbb{R}$ is called a λ-*set* if each countable subset $A \subset X$ is G_δ in $\mathbb{R}$.

Pseudocompleteness for space of Baire-one functions

Recall that a topological space X is called a λ-*space* if every countable subset is of type G_δ in X. A subset X of the real line $\mathbb{R}$ is called a λ-*set* if each countable subset $A \subset X$ is G_δ in $\mathbb{R}$.

Theorem. (T. Banakh, S. Gabriyelyan, 2020)

Let X be a space of countable pseudocharacter. A space $B_1(X)$ is Choquet if and only if X is a λ-space.

Corollary 4

Let X be a space of countable pseudocharacter. A space $B_1(X)$ is pseudocomplete if and only if X is a λ-space.

Let us recall the definition of some small uncountable cardinal.
$\mathfrak{b} = \min\{|X| : X$ has a countable pseudocharacter but X is not a λ-space$\}$.

Proposition 1

Let X be a space of countable pseudocharacter. If $|X| < \mathfrak{b}$ then $B_1(X)$ is Choquet (pseudocomplete).

Example 1.

It is consistent with *ZFC* there is a zero-dimensional metrizable separable space X such that

1. $B_1(X)$ is not pseudocomplete;
2. $B_1(X)$ is Baire;
3. $|X| = \mathfrak{b}$;
4. X is a γ-space;
5. X is not a λ-space.

Let us give several examples of subsets X of the real line $\mathbb{R}$ for which $B_1(X)$ is Baire.

ZFC Examples

(1) $B_1(\mathbb{Q})$, where $\mathbb{Q}$ is the space of all rational numbers (or any countable subset of $\mathbb{R}$), is Baire. This is because $\mathfrak{b}$ is uncountable.

(2) Let X be an uncountable λ-set that is a subset of the real line. Then $B_1(X)$ is Baire.

Consistent Examples

(3) For any uncountable subset X of the real line of cardinality $< \mathfrak{b}$, $B_1(X)$ is Baire.

(4) Let X be an uncountable γ-set that is a subset of the real line. Then $B_1(X)$ is Baire.

References

Osipov A.V. Baire property of space of Baire-one functions. European Journal of Mathematics, 11:8 (2025).

• A space X is *Fréchet-Urysohn* if, for any $A \subseteq X$ and any $x \in \overline{A}$, there exists a sequence $\{a_n : n \in \mathbb{N}\} \subseteq A$ that converges to x.

• A space X is called *k-Fréchet-Urysohn* if, for any open set $U \subset X$ and any point $x \in \overline{U}$, there exists a sequence $\{x_n : n \in \mathbb{N}\} \subset U$ that converges to x. Clearly, every Fréchet-Urysohn space is k-Fréchet-Urysohn.

The class of k-Fréchet-Urysohn spaces was introduced by Arhangel'skii but, long before that, Mrówka proved (without using the term) that any product of first countable spaces is k-Fréchet-Urysohn.

- A space X is said to have *property* (κ) if every pairwise disjoint sequence of finite subsets of X has a strongly point-finite subsequence.

A family $\{A_\alpha : \alpha \in \kappa\}$ of subsets of a set X is said to be *point-finite* if for every $x \in X$, $\{\alpha \in \kappa : x \in A_\alpha\}$ is finite.
A family $\{A_\alpha : \alpha \in \kappa\}$ of subsets of a space X is said to be *strongly point-finite* if for every $\alpha \in \kappa$, there exists an open set U_α of X such that $A_\alpha \subset U_\alpha$ and $\{U_\alpha : \alpha \in \kappa\}$ is point-finite.

- A space X has *the Banakh property* if there is a countable family $\{A_n : n \in \mathbb{N}\}$ of closed nowhere dense subsets of X such that for any compact subset K of X, there is $n \in \mathbb{N}$ with $K \subseteq A_n$.

- Denote by $C_k(X)$ the space $C(X)$ of all real-valued continuous functions on X endowed with the compact-open topology. X is called an *Ascoli space* if every compact subset K of $C_k(X)$ is evenly continuous (i.e., if the map $(f, x) \longmapsto f(x)$ is continuous as a map from $K \times X$ to $\mathbb{R}$).

C_p-theory, property (κ)

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- (Gabriyelyan) The κ-Fréchet-Urysohn property of $C_p(X)$ implies that $C_p(X)$ is Ascoli.

- (Tkachuk) $C_p(X)$ is κ-Fréchet-Urysohn if and only $C_p(X, [0, 1])$ is κ-Fréchet-Urysohn.

Theorem 8.

For any space X, the following conditions are equivalent:

1. *The space X has the property (κ).*
2. *The space $C_p(X)$ is κ-Fréchet-Urysohn.*
3. *The space $C_p(X, [0, 1])$ is κ-Fréchet-Urysohn.*
4. *The space $C_p(X)$ is Ascoli.*
5. *The space $C_p(X)$ does not have the Banakh property.*

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5. *The space $C_p(X)$ does not have the Banakh property.*
6. *The space $B_1(X)$ is Baire.*

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1. *The space X has the property (κ).*
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3. *The space $C_p(X, [0, 1])$ is κ-Fréchet-Urysohn.*
4. *The space $C_p(X)$ is Ascoli.*
5. *The space $C_p(X)$ does not have the Banakh property.*
6. *The space $B_1(X)$ is Baire.*
7. *Every pairwise disjoint sequence of non-empty finite subsets of X has a strongly G_δ-disjoint subsequence.*

References

Osipov A.V. The κ-Fréchet-Urysohn property for $C_p(X)$ is equivalent to Baireness of $B_1(X)$. Mathematica Slovaca (2025).

- A family $\{A_\alpha : \alpha \in \kappa\}$ of subsets of a set X is said to be *point-finite* if for every $x \in X$, $\{\alpha \in \kappa : x \in A_\alpha\}$ is finite.

Kąkol, Kurka, Leiderman

A space X has the Δ_1-*property* if any disjoint sequence $\{A_n : n \in \omega\}$ of countable subsets of X has a point-finite open expansion, i.e., there exists a point-finite family $\{U_n : n \in \omega\}$ of open subsets of X such that $A_n \subset U_n$ for every $n \in \omega$. The spaces with the Δ_1-property are also called Δ_1-spaces.

Below we mention three facts justifying the motivation of our interest.
(1) The class Δ_1 contains properly the subclass consisting of all λ-spaces. Also, a subset $X \subseteq \mathbb{R}$ is Δ_1 if and only if X is a λ-set. We argue that the class Δ_1 provides a natural extension of the family of all λ-sets beyond the separable metrizable spaces.

(2) Any $X \in \Delta_1$ has property (κ) which is equivalent to the property that $C_p(X)$ is κ-Frechet-Urysohn, by the theorem of M. Sakai.

(3) The class Δ_1 is tightly connected to the study of binormality in non-separable Banach spaces.

Theorem.

A space X has the Δ_1-property if and only if, for any function $f \in \mathbb{R}^X$ such that $f^{-1}(\mathbb{R} \setminus \{0\})$ is countable, there exists a pointwise bounded set $B \subset Cp(X)$ such that $f \in \overline{B}$.

Theorem 9.

For a space X, the following conditions are equivalent:

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For a space X, the following conditions are equivalent:

1. X has the Δ_1-property.
2. $B_1(X)$ is Choquet.
3. $B_1(X)$ is pseudocomplete.
4. Every countable subset of X is strongly Coz_δ-disjoint.

Solution of open questions

Definition.

Let $\mathcal{S} = \{S_n : n \in \omega\}$ be a family of subsets of a space X and $x \in X$. Then $\mathcal{S}$ *weakly converges* to x if for every neighborhood W of x there is a sequence $(s_n : n \in \omega)$ such that $s_n \in S_n$ for each $n \in \omega$ and there is n' such that $s_n \in W$ for each $n > n'$.

Proposition.(Kakol, Leiderman, Tkachuk, 2024)

If X has no weakly convergent sequences. Then X has the property (κ).

Solution of open questions

(A. Lipin) If X is a crowded submaximal space, then it has no weakly convergent sequences.

Corollary

If X is a crowded submaximal space, then X has the property (κ).

This answers in the positive Question 4.10 of (KLT).

Corollary

If X is a crowded submaximal space, then $B_1(X)$ is Baire.

• (Juhász, L. Soukup, Z. Szentmiklóssy), it is proved under *ZFC* that for each infinite cardinal κ the Cantor cube D^{2^κ} contains a dense submaximal subspace X with $|X| = \Delta(X) = \kappa$.

Thus in the case $k = \omega$ there is a countable dense subset in the Cantor cube $D^{\mathfrak{c}}$ which has no weakly convergent sequences.

This answers in the positive Question 4.8 of (KLT).

Theorem 10.

There exists in *ZFC* a separable pseudocompact dense subspace $P \subset D^{\mathfrak{c}}$ which is not a Δ_1-space but has the property (κ).

This answers in the positive Question 4.7 of (KLT).

Corollary.

There exists in *ZFC* a separable pseudocompact dense subspace $P \subset D^{\mathfrak{c}}$ such that $B_1(P)$ is Baire, but $B_1(P)$ is not Choquet.

References

Osipov A.V. The Δ_1-property of X is equivalent to the Choquet propertys of $B_1(X)$. Topology and its Appl. 370 (2025), 109395.

$B_1(X,[0,1])$, $B_1(X,\{0,2\})$, ... ?

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$$
\begin{tikzcd}
B_1(X,\mathbf{R}) \text{ — Шоке} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & B_1(X,\mathbf{R}) \text{ — бэровское} \arrow[d, Rightarrow] \\
B_1(X,[0;1]) \text{ — Шоке} \arrow[r, Rightarrow] & B_1(X,[0;1]) \text{ — бэровское}
\end{tikzcd}
$$