In memory of the victims of the earthquake

2nd International E-Conference on Mathematical and Statistical Sciences:
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Representations of flat virtual braid groups by automorphisms of free groups

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Plan of the talk

1. **Braid Groups**
2. **Virtual Braid Groups**
3. **Flat Virtual Braid Groups**
4. **Main Results**

Braid Groups

Artin braid groups

E. Artin [1925] defined a braid group B_n, $n \geq 1$, as follows.

Generators:

$$\sigma_1, \sigma_2, \ldots, \sigma_{n-1}.$$

Braid Relations:

$$\sigma_i \, \sigma_{i+1} \, \sigma_i = \sigma_{i+1} \, \sigma_i \, \sigma_{i+1}, \qquad i = 1, \ldots, n-2; \tag{1}$$

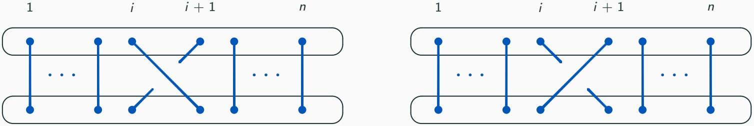
$$\sigma_i \, \sigma_j = \sigma_j \, \sigma_i, \qquad |i-j| \geq 2. \tag{2}$$

Denote by $i_n : B_n \to B_{n+1}$ an injective homomorphism such that $\sigma_i \in B_n$ goes to $i_n(\sigma_i) = \sigma_i \in B_{n+1}$ for $i = 1, \ldots, n-1$. It induces the sequence of group inclusions

$$B_1 \subset B_2 \subset \ldots \subset B_n \subset \ldots$$

Geometric braids

Geometric generators of the braid group B_n on n strands in $\mathbb{R}^3$:



σ_i σ_i^{-1}

Let Σ_n be the symmetric group on n symbols $\{1, \ldots, n\}$. Consider transpositions $s_i = (i, i+1) \in \Sigma_n$, where $i = 1, \ldots, n-1$.

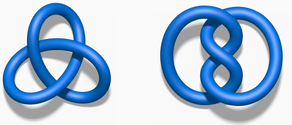
Remark. Transpositions $s_1, s_2, \ldots, s_{n-1}$ satisfy braid relations.

There is the unique homomorphism $\pi_n : B_n \to \Sigma_n$ such that $\pi_n(\sigma_i) = s_i$ for $i = 1, \ldots, n-1$. Since Σ_n is generated by transpositions $s_1, s_2, \ldots, s_{n-1}$, the homomorphism π_n is surjective.

The kernel $P_n = \mathrm{Ker}(\pi_n)$ is known as the pure braid group.

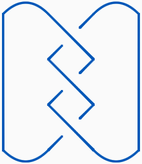
Braids and Knots

A knot K is a smooth embedding of S^1 into $\mathbb{R}^3$. Two knots K_1 and K_2 are said to be equivalent if there is an isotopy of $\mathbb{R}^3$ taking K_1 to K_2.



Theorem. [J. Alexander, 1923] Every knot in S^3 can be represented as a closure of a braid.

For any braid $\beta \in B_n$ denote it's closure by $\hat{\beta}$.

$\beta = \sigma_1^3 \in B_2$  $\hat{\beta}$ is the trefoil knot 

Automorphism group of a free group

Let $\mathbb{F}_n$ be a free group of rank n with free generators $\{x_1, \ldots, x_n\}$, and $\mathrm{Aut}(\mathbb{F}_n)$ be the automorphism group of $\mathbb{F}_n$.

Theorem. [E. Artin, 1925] There is an embedding $\varphi_n : B_n \to \mathrm{Aut}(\mathbb{F}_n)$ of the braid group B_n given by

$$\varphi_n(\sigma_i) : \begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1}, \\ x_j \mapsto x_j, \quad j \neq i, i+1. \end{cases}$$

Braid groups and mapping class groups

Let $S = S_{g,b,n}$ denote a 2-manifold of genus g with b boundary components and n punctures, and let $\mathrm{Diff}^+(S)$ denote the group of all orientation preserving diffeomorphisms of S.

Let assign the compact–open topology to $\mathrm{Diff}^+(S)$ making it into a topological group.

The mapping class group $\mathcal{M} = \mathcal{M}_{g,b,n}$ of S is $\pi_0(\mathrm{Diff}^+(S))$, that is, the quotient of $\mathrm{Diff}^+(S)$ modulo its subgroup of all diffeomorphisms of S which are isotopic to the identity rel ∂S. We allow diffeomorphisms in $\mathrm{Diff}^+(S)$ to permute the punctures.

Theorem.[1] There is a natural isomorphism

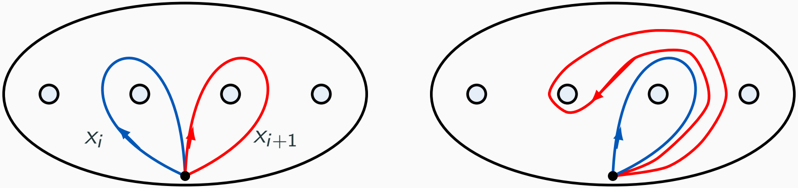
$$B_n \cong \mathcal{M}_{0,1,n}.$$

[1] J. Birman, Braids, links and mapping class groups. 1974.

Disc with punctures

Let $S = S_{0,1,n}$ be a disc with n punctures, $\pi_1(S) \cong \mathbb{F}_n = \langle x_1, \dots, x_n \rangle$.

Generator σ_i induces the following geometric action on generators x_i and x_{i+1} of the fundamental group $\pi_1(S)$:



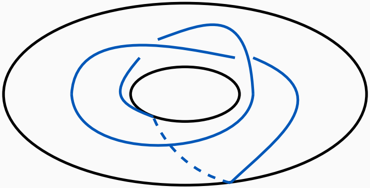
$$\varphi_n(\sigma_i) : \begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1}, \\ x_j \mapsto x_j, \quad j \neq i, i+1. \end{cases}$$

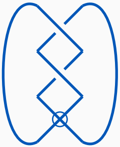
Action on the boundary ∂S is identity: $\varphi_n(\sigma) : x_1 x_2 \dots x_n \to x_1 x_2 \dots x_n$.

Virtual Braid Groups

Virtual diagrams and virtual knots

A virtual knot is a smooth embedding of S^1 into handlebody H_g of some genus g. Two virtual knots K_1 and K_2 are said to be equivalent if there is an isotopy of H_g and stabilizations/destabilizations of the handlebody that send K_1 to K_2.





A virtual knot can have virtual crossings in a projection on a plane if the knot goes along a handle. A virtual crossing is marked by a small circle.

Virtual braid group

L. Kauffman [1999] defined virtual braid groups VB_n, $n \geq 2$, with generators $\sigma_1, \ldots, \sigma_{n-1}, \rho_1, \ldots, \rho_{n-1}$ and the following defining relations:

- relations for classical and virtual generators:

$$
\begin{array}{ccc}
 & \rho_i^2 = 1, & 1 \leq i \leq n-1, \\
\sigma_i \, \sigma_{i+1} \, \sigma_i = \sigma_{i+1} \, \sigma_i \, \sigma_{i+1}, & \rho_i \, \rho_{i+1} \, \rho_i = \rho_{i+1} \, \rho_i \, \rho_{i+1}, & 1 \leq i \leq n-2, \\
\sigma_i \, \sigma_j = \sigma_j \, \sigma_i, & \rho_i \, \rho_j = \rho_j \, \rho_i, & |i-j| \geq 2.
\end{array}
$$

- mixed relations:

$$
\begin{array}{cc}
\rho_i \, \rho_{i+1} \, \sigma_i = \sigma_{i+1} \, \rho_i \, \rho_{i+1}, & 1 \leq i \leq n-2, \\
\rho_i \, \sigma_j = \sigma_j \, \rho_i, & |i-j| \geq 2.
\end{array}
$$

Theorem.[2] Any virtual link can be presented as a closed virtual braid.

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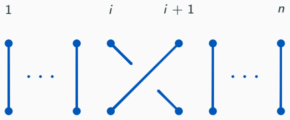
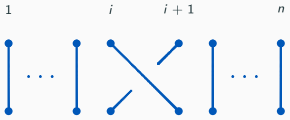
[2] S. Kamada, *Braid presentation of virtual knots and welded knots*, Osaka J. Math., **44** (2007), 441–458.

Geometric generators

Classical generators:

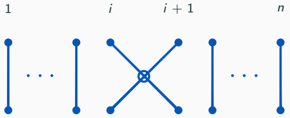


σ_i



σ_i^{-1}

Virtual generators:



ρ_i

Representation of virtual braid groups by automorphisms

Representations $VB_n \to \mathrm{Aut}(G_n)$, where G_n is a group, were studied intensively in last years. It was done for the following groups G_n:

- $G_n = \mathbb{F}_n * \mathbb{Z}^{n+1}$, by D. Silver, S. Williams [2001];
- $G_n = \mathbb{F}_n$, by H. Boden, E. Dies, A. Gaudreau, A. Gerlings, E. Harper, A. Nikas [2015];
- $G_n = \mathbb{F}_n * \mathbb{Z}^{2n+1}$ and $G_n = \mathbb{F}_n * \mathbb{Z}^n$, by V. Bardakov, Yu. Mihalchishina, M. Neshchadim [2017].

All the constructed representations are not faithful.

Flat Virtual Braid Groups

Flat virtual braid group

L. Kauffman [2000] introduced flat virtual braid groups FVB_n, $n \geq 2$, with generators $\sigma_1, \ldots, \sigma_{n-1}, \rho_1, \ldots, \rho_{n-1}$ and the following defining relations:

- relations for classical and virtual generators:

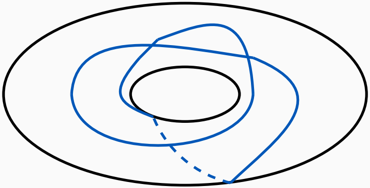
$$\begin{array}{ccc} \sigma_i^2 = 1, & \rho_i^2 = 1, & 1 \leq i \leq n-1, \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, & \rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1}, & 1 \leq i \leq n-2, \\ \sigma_i \sigma_j = \sigma_j \sigma_i, & \rho_i \rho_j = \rho_j \rho_i, & |i-j| \geq 2. \end{array}$$

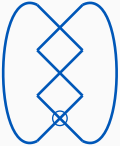
- mixed relations:

$$\begin{array}{cc} \rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1}, & 1 \leq i \leq n-2; \\ \rho_i \sigma_j = \sigma_j \rho_i, & |i-j| \geq 2. \end{array}$$

Flat virtual knots

From now we will not distinguish over/under crossings. Consider a virtual analog of the trefoil knot.



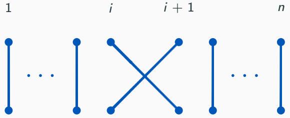


Then we will get a collection of curves on a handlebody. An equivalence of two curves is defined similar to the equivalence of virtual knots.

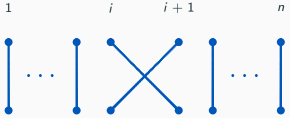
The class of equivalent curves is called a flat virtual link, in particular we get a flat virtual knot in the case of one curve on a handlebody.

Geometric generators

Flat classical generators, $\sigma_i^{-1} = \sigma_i$:

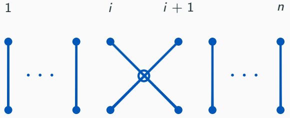


σ_i



σ_i^{-1}

Virtual generators:



ρ_i

Forbidden relations

M. Goussarov, M. Polyak, O. Viro [2000] demonstrated that the following relations do not hold in VB_n (and so, in FVB_n):

$$\rho_i \, \sigma_{i+1} \, \sigma_i = \sigma_{i+1} \, \sigma_i \, \rho_{i+1}, \qquad 1 \leq i \leq n-2,$$
$$\rho_{i+1} \, \sigma_i \, \sigma_{i+1} = \sigma_i \, \sigma_{i+1} \, \rho_i, \qquad 1 \leq i \leq n-2.$$

These relations are called forbidden relations.

Problem.[3] Does there exist a representation of FVB_n by automorphisms of some group such that the forbidden relations do not hold?

[3] R. Fenn, D. Ilyutko, L. Kauffman, V. Manturov, *Unsolved problems in virtual knot theory and combinatorial knot theory*, Banach Center Publications, **103** (2014), 9–61.

Main Results

Flat virtual braids and free groups

Consider the free group $\mathbb{F}_{2n} = \langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle$ of rank $2n$.

Let $\Theta_n : FVB_n \to \mathrm{Aut}(\mathbb{F}_{2n})$ be a correspondence which sent generators $\sigma_i, \rho_i \in FVB_n$ to the following automorphisms:

$$\Theta_n(\sigma_i) : \begin{cases} x_i \mapsto x_{i+1}\, a_i(y_i, y_{i+1}), \\ x_{i+1} \mapsto x_i\, b_i(y_i, y_{i+1}), \end{cases} \qquad \Theta_n(\rho_i) : \begin{cases} x_i \mapsto x_{i+1}\, c_i(y_i, y_{i+1}), \\ x_{i+1} \mapsto x_i\, d_i(y_i, y_{i+1}), \\ y_i \mapsto y_{i+1}, \\ y_{i+1} \mapsto y_i, \end{cases}$$

where elements $a_i(y_i, y_{i+1})$, $b_i(y_i, y_{i+1})$, $c_i(y_i, y_{i+1})$ and $d_i(y_i, y_{i+1})$ are words in the free group $\mathbb{F}_2$ of rank 2 with free generators $\{y_i, y_{i+1}\}$, where $1 \leq i \leq n-1$.

Question. For which words $a_i(y_i, y_{i+1})$, $b_i(y_i, y_{i+1})$, $c_i(y_i, y_{i+1})$ and $d_i(y_i, y_{i+1})$ the correspondence Θ_n is homomorphism?

The infinite family of local representations

Theorem 1. [V. – Chuzhinov, 2023]

The correspondence $\Theta_n : FVB_n \to \mathrm{Aut}(\mathbb{F}_{2n})$ is homomorphism if and only if

$$b_i(y_i, y_{i+1}) = a_i^{-1}(y_i, y_{i+1}), \quad c_i(y_i, y_{i+1}) = y_{i+1}^{m_i}, \quad d_i(y_i, y_{i+1}) = y_i^{-m_i},$$

where $m_i \in \mathbb{Z}$ for $1 \leq i \leq n-1$ and

$$a_j(y_j, y_{j+1}) = y_{j+1}^{m_j}\, a_{j-1}(y_j, y_{j+1})\, y_j^{-m_{j-1}}, \qquad 2 \leq j \leq n-1,$$

for $n \geq 3$, where $a_1 = w(y_1, y_2)$ for some word $w(A, B) \in \mathbb{F}_2 = \langle A, B \rangle$. Moreover, Θ_n does not preserve forbidden relations.

Representations Θ_n are local in the following sense: $\Theta_n(\sigma_i)$ and $\Theta_n(\rho_i)$ act trivially on x_j and y_j for $j \neq i, i+1$.

Automorphisms corresponding to classical generators

Since a representation Θ_n depends on a word $w \in \mathbb{F}_2$ and a vector of integers $m = (m_1, \ldots, m_{n-1})$, we denote it by $\Theta_n^{w,m}$. Then

$$\Theta_n^{w,m}(\sigma_1) : \begin{cases} x_1 \mapsto x_2\, w(y_1, y_2), \\ x_2 \mapsto x_1\, (w(y_1, y_2))^{-1}, \end{cases}$$

and

$$\Theta_n^{w,m}(\sigma_i) : \begin{cases} x_i \mapsto x_{i+1} \displaystyle\prod_{k=i}^{2} y_{i+1}^{m_k}\, w(y_i, y_{i+1}) \prod_{k=1}^{i-1} y_i^{-m_k}, \\ x_{i+1} \mapsto x_i \displaystyle\prod_{k=i-1}^{1} y_i^{m_k}\, (w(y_i, y_{i+1}))^{-1} \prod_{k=2}^{i} y_{i+1}^{-m_k}, \end{cases}$$

for $i \geq 2$, where in the products $\prod_{k=i}^{2}$ and $\prod_{k=i-1}^{1}$ indices decrease, as well as in the products $\prod_{k=1}^{i-1}$ and $\prod_{k=2}^{i}$ indices increase.

The word w is called the defining word for the homomorphism $\Theta_n^{w,m}$.

Automorphisms corresponding to virtual generators

For the virtual generators ρ_i the automorphisms of the free group $\mathbb{F}_{2n}$ with free generators $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ are defined by

$$\Theta_n^{w,m}(\rho_i) : \begin{cases} x_i \mapsto x_{i+1} y_{i+1}^{m_i}, \\ x_{i+1} \mapsto x_i y_i^{-m_i}, \\ y_i \mapsto y_{i+1}, \\ y_{i+1} \mapsto y_i. \end{cases}$$

In general, the constructed homomorphisms $\Theta_n^{w,m}$ are not faithful.

Question. What is the kernel $\mathrm{Ker}(\Theta_n^{w,m})$?

Pure and Kure flat virtual braid groups

Let Σ_n be the symmetry group generated by transpositions $\{s_1, \ldots, s_{n-1}\}$ on $\{1, \ldots, n\}$, where $s_i = (i, i+1)$.

Consider subgroup $S_n = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle < FVB_n$ isomorphic to Σ_n. Let $\pi_n : FVB_n \to S_n$ be a homomorphism defined on generators:

$$\pi_n(\sigma_i) = \sigma_i, \qquad \pi_n(\rho_i) = \sigma_i, \qquad 1 \le i \le n-1.$$

The kernel $FVP_n = \mathrm{Ker}(\pi_n)$ is known as the flat virtual pure braid group.

Consider subgroup $S'_n = \langle \rho_1, \ldots, \rho_{n-1} \rangle < FVB_n$ isomorphic to Σ_n. Let $\nu_n : FVB_n \to S'_n$ be a homomorphism defined on generators:

$$\pi_n(\sigma_i) = 1, \qquad \pi_n(\rho_i) = \rho_i, \qquad 1 \le i \le n-1.$$

The kernel $KFB_n = \mathrm{Ker}(\nu_n)$ is known as the kure flat virtual braid group.

Lemma. Let $n \geq 2$. For any defining word $w = w(A, B) \in \mathbb{F}_2$ and any vector of integers $m = (m_1, \ldots, m_n)$ the following property holds:

$$\mathrm{Ker}(\Theta_n^{w,m}) \leq FVP_n \cap KFB_n.$$

Theorem 2. [V.–Chuzhinov, 2023]

The presentation $\Theta_2^{w,m} : FVB_2 \to \mathrm{Aut}(\mathbb{F}_4)$ with $m = (m_1)$ is not faithful if and only if the defining word $w \in \langle A, B \rangle$ is of the form

$$w(A, B) = A^{k_1} B^{k_2} \ldots A^{k_p} B^{-k_p} \ldots A^{-k_2} B^{-k_1} A^{m_1},$$

where k_i are integers with $k_i \neq 0$ for $i = 2, \ldots, p-1$.

Moreover, in this case $\mathrm{Ker}(\Theta_2^{w,m}) = FVP_2 \cap KFB_2$.

The case $n \geq 3$

Theorem 3. [V.–Chuzhinov, 2023]

Let $n \geq 3$. For any defining word $w = w(A, B) \in \mathbb{F}_2$ and any vector of integers $m = (m_1, \ldots, m_{n-1})$ the kernel $\mathrm{Ker}(\Theta_n^{w,m})$ contains a subgroup isomorphic $\mathbb{F}_2$.

For $n \geq 3$ from the above results we get:

$$\mathbb{F}_2 < \mathrm{Ker}(\Theta_n^{w,m}) < FVP_n \cap KFB_n$$

and

$$\mathbb{F}_3 * \mathbb{Z}^2 < FVP_n \cap KFB_n.$$

Open Problems:

- Find $FVP_n \cap KFB_n$ for $n \geq 2$;
- Find $\mathrm{Ker}(\Theta_n^{w,m})$ for $n \geq 3$.

Thank you!

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